# ABSTRACTS OF PAPERS

## SUBMITTED FOR PRESENTATION TO THE SOCIETY

The following papers have been submitted to the Secretary and the Associate Secretaries of the Society for presentation at meetings of the Society. They are numbered serially throughout this volume. Cross references to them in the reports of the meetings will give the number of this volume, the number of this issue, and the serial number of the abstract.

### 406. H. W. Brinkmann: *Relations between zeta-functions of different algebraic fields.*

Let $k_1, k_2, \cdots, k_\nu$ be algebraic fields and $\zeta_i(s)$ the zeta-function of $k_i$. This paper deals with relations of the form $\zeta_1^{a_1}\zeta_2^{a_2}\cdots\zeta_\nu^{a_\nu}=1$, where $a_i$ are integers (positive or negative). It is shown how, for given fields $k_i$, all such relations can be found. Several applications are given. (Received September 27, 1938.)

### 407. Richard Courant: *Conformal mapping of non-orientable surfaces on plane domains.*

Riemannian manifolds with finite characteristic number but possibly with infinitely many boundaries can be mapped conformally on a plane slit domain so that cross-caps or handles are represented by proper coordination of edges of slits in the plane. This result has been applied by the author to the solution of Plateau's problem for arbitrary topological structure. The present paper is an extension to one-sided surfaces of former results obtained for orientable surfaces. The method consists in a topological discussion of stream lines of an analytic function on the surface with one pole. A detailed proof is contained in a paper on conformal mapping to appear in the American Journal of Mathematics. (Received September 24, 1938.)

### 408. Richard Courant: *New remarks on Plateau's problem.*

The problem of the existence of minimal surfaces of given topological structure under general boundary conditions is solved by new devices on the basis of a variational problem of the Dirichlet type. In the case of genus zero the parameter domain is a Riemann surface consisting of $k$ unit circles connected in $2k-2$ branch points. For higher genus a corresponding number of full planes are attached in 4 branch points each. After the variational problem has been solved, the character of the solution as a minimal surface is an immediate consequence of the variational conditions for the $k$ boundary circles and the branch points. The possibility of conformal mapping of a $k$-fold connected domain on a $k$-fold unit circle with $2k-2$ branch points is a side result. For higher topological structure, the intrinsic advantage of using mapping theorems is discussed. (Received September 27, 1938.)

### 409. F. A. Ficken: *The Riemannian and affine differential geometry of product-spaces.*

The object of this paper is to discuss systematically the Riemannian (or affine) differential geometry of the topological product of two or more Riemann (or affinely connected) spaces. In the direct (orthogonal) product of two Riemann spaces, product-tensors and product-connections are defined and some of their properties are given. After several general theorems are proved, results are presented on the following topics: geodesic properties of subspaces, parallel displacement and curvature, parallel fields of vector spaces, and motions. Certain of these theorems are then extended to the direct (affine) product of two affinely connected manifolds. Returning to the product of two Riemann spaces, which is no longer assumed to be direct, after proving several general theorems the author shows, finally, that if such a space has any one of a number of simple properties, it is necessarily the direct product of its factors. (Received September 19, 1938.)

### 410. Tomlinson Fort: *The Euler-Maclaurin summation formula.*

The classical Euler-Maclaurin sum formula expresses a sum as an integral plus terms involving the Bernoulli numbers. The present paper gives an extension to $n$-fold sums which are expressed as integrals plus terms involving the Bernoulli numbers of higher order as defined by Nörlund. If $n=1$, the formula reduces to the classical formula and the remainder formula developed in the paper reduces to the classical Jacobi form. (Received September 28, 1938.)

### 411. D. W. Hall and A. D. Wallace: *Some invariants under monotone transformations.*

Let $A$ be a locally connected continuum and let $P$ be a property of point sets. If $P$ is the property of being connected, it has been shown (Kuratowski, Fundamenta Mathematicae, vols. 8 and 13) that the following properties are equivalent. $\Delta_0(P)$: If $A$ is the sum of two continua, their product has property $P$. $\Delta_1(P)$: The boundary of a component of the complement of a continuum has property $P$. $\Delta_2(P)$: Each irreducible separation of $A$ between two points has property $P$. $\Delta_3(P)$: If the sets $X$ and $Y$ are disjoint closed sets containing the points $x$ and $y$, then there exists a closed set having property $P$ that separates $A$ between $x$ and $y$ and is disjoint with both $X$ and $Y$. In this note, the following theorems are proved: (1) for each $i=0, 1, 2$ and any property $P$ of point sets, $\Delta_i(P)$ implies $\Delta_{i+1}(P)$. (2) If the property $P$ is invariant under monotone transformations, so also is $\Delta_i(P)$. (Received September 29, 1938.)

### 412. T. R. Hollcroft: *Curve systems with distinct nodes and cusps and of negative virtual dimension.*

Plane sections of tangent cones to a family of singular surfaces in $S_3$ with only a cuspidal curve in common constitute an irregular plane curve system whose only singularities are distinct nodes and cusps. This system is defined by two equations in one parameter. The virtual dimension of this system is negative for high orders of the surface. In $S_r$ the tangent cone to a nonsingular primal $V_{r-1}$ of order $\nu$ from an $S_{r-2}$ cuts $V_{r-1}$ in the contour curves $\Gamma$. From $S_{r-2}$, the system $\Gamma$ projects into a system of plane curves $C$ which is irregular for the following limiting values of $r$ and $\nu$: $r=3$, $\nu\geqq 5$; $r\geqq 4$, $\nu\geqq 3$. This system $C$ is defined by a set of three equations in two parameters. Its only singularities are distinct nodes and cusps. The system $C$ has a negative virtual dimension for the following limiting values of $r$ and $\nu$: $r=4$, $\nu\geqq 8$; $r=5$, $\nu\geqq 5$; $r=6, 7, 8$, $\nu\geqq 4$; $r\geqq 9$, $\nu\geqq 3$. The above systems may in some cases be reducible. (Received October 1, 1938.)

**413. A. N. Lowan:** *On Green's functions in the theory of heat conduction in spherical coordinates.*

This paper is a sequel to the paper *On the operational determination of two dimensional Green's functions in the theory of heat conduction* (this Bulletin, vol. 44 (1938), pp. 125–133). Expressions for Green's functions for a sphere, and for the infinite solid bounded internally by a sphere, are obtained. The corresponding Laplace transforms are derived in a form which makes it possible to use the results of the previous paper. The inversion of these transforms is then accomplished as before. (Received September 19, 1938.)

**414. A. N. Lowan:** *On the computation of the second difference of the $Si(x)$, $Ci(x)$, and $Ei(x)$ functions.*

(1) Define $R(x) = [\phi(x+h)+\phi(x-h)-2\phi(x)]-[(h/2)\phi'(x+h)-(h/2)\phi'(x-h)]$; whence (2) $|R(x)<\sum_{k=2}^{\infty}[h^{2k}/(2k-1)!]\{\phi^{(2k)}(x)\}$ where $\phi(x)$ is any of the above functions and $\{\phi^{(2k)}(x)\}$ an upper bound of the modulus of its $2k$th derivative. The latter is expressed as a definite integral, and its upper bound is found. For $x$ larger than a certain $x_0$ and $h=10^{-4}$, it is shown that the upper bounds of $R(x)$ for each of the functions under consideration is of the order of magnitude of 5 times $10^{-12}$. The first bracket in (1), which represents the second difference of $\phi(x)$, may therefore be approximated by the second term in (1), with an error not exceeding 5 times $10^{-12}$. This result is applied as a check on the computation of $Si(x)$, $Ci(x)$, and $Ei(x)$, for the range between 0 and 2 at intervals of $10^{-4}$, in progress now by the W.P.A. Project for the Computation of Mathematical Tables, sponsored by Dr. Lyman J. Briggs, Director of the National Bureau of Standards, Washington, D. C. (Received September 19, 1938.)

**415. J. D. Mancill:** *Problems of the calculus of variations with prescribed transversality conditions.*

Problems of the calculus of variations in $(x, y_1, y_2, \cdots, y_n)$ space for which a prescribed relation exists between the directions of the extremals and the transversal directions were studied first by Rawles (Transactions of this Society, vol. 30 (1928), pp. 765–784). More recently, LaPaz, using a method and point of view quite different from that of Rawles, has given a rather complete treatment of the problem in nonparametric form (this Bulletin, vol. 36 (1930), pp. 674–680). In the present paper, a method similar to that of LaPaz is used, which avoids, however, his very intricate treatment of an associated system of nonhomogeneous partial differential equations by reducing the problem to a very simple total differential equation. The method applies with equal facility to parametric and non-parametric problems of the calculus of variations in space of any number of dimensions. The problem is treated in parametric form. Necessary and sufficient conditions in order that a transversality relation belong to a problem of the calculus of variations are derived. Finally, there is obtained the most general integrand function of a problem of the calculus of variations to which a given transversality relation belongs. (Received September 2, 1938.)

**416. J. A. Shohat:** *On the generalized orthogonal polynomials.*

Consider a complete (that is, of all possible degrees $n=0, 1, 2, \cdots$) sequence of polynomials $\{\Phi_n(x)\}$ with highest coefficient unity. It is shown, on the basis of a recently established theorem of R. P. Boas, that a necessary and sufficient condition for the said sequence to form an orthogonal system with respect to infinitely many weight functions $\psi(x)$ of bounded variation in $(-\infty, \infty)$ is: $\phi_{n+2}(x)=(x-c_{n+2})\Phi_{n+1}(x)-\lambda_{n+2}\Phi_n(x)$, $(n=0, 1, \cdots)$, with all $\lambda_n$ different from zero. For $\lambda_n>0$, $\{\Phi_n(x)\}$ is an ordinary sequence of orthogonal polynomials, that is, the weight function $\psi(x)$ is monotone and non-decreasing in $(-\infty, \infty)$. (Received September 29, 1938.)

### 417. Louis Weisner: *A characteristic property of completely decomposable groups.*

A completely decomposable group is one which is the direct product of simple groups. The author proves that a finite composite group is completely decomposable if, and only if, the cross cut of its maximal invariant subgroups is the identity. (Received September 2, 1938.)

### 418. H. A. Arnold: *Topological order in Kantorovitch spaces.*

Consider a partially ordered space $S$ of Kantorovitch with elements $x, y, z, \cdots$. Given a completely continuous transformation $F(x)$ transforming the closure $\overline{W}$ of an open bounded subset $W$ into a subset of $S$, there exists a function $F_\epsilon(x)$ and a fixed $y$ in $S$ such that $|F(x)-F_\epsilon(x)|<\epsilon y$ for all $x$ in $\overline{W}$, and such that the values of $F(x)$ are all in a fixed finite-dimensional linear subspace of $S$. The definitions of topological degree and index follow readily. It is easily proved that compact sets are bounded. Necessary and sufficient conditions are derived, in lattice language, that the space be bicompact; that for every sequence $\{p_i\}$ such that $p_i\rightarrow 0$ we have $|p_i|<\epsilon y$ for $i>N$ and $y$ fixed throughout; that to a given compact set there exist a fixed $y$ such that to any $\epsilon$ there are elements $x_i$ finite in number with the property that, for every other element $x$ of the set, $|x-x_i|<\epsilon y$ for at least one $i$. Necessary and sufficient conditions are derived that these properties be pairwise equivalent. (Received October 29, 1938.)

### 419. Clifford Bell: *Plane curves with pseudo rhamphoid cusps.*

In a previous paper (American Mathematical Monthly, vol. 64 (1937), pp. 218–221) the author developed methods of determining the parameters of the cusps and of those points for which the tangent line has more than two point contact with the curve. The parametric equations used were assumed to give a proper parametric representation of the curve. In this paper those curves are considered whose parametric equations improperly represent other curves. Consider a curve as being traced out by a moving point whose motion is governed by the parameter. A pseudo cusp on such a curve is defined as a point at which the moving point stops, reverses its direction, and then moves back along the path upon which it arrived. Such pseudo cusps, together with the ordinary cusps, are obtained by the methods of the previous paper when applied to the above mentioned curves, whose parametric equations improperly represent other curves. (Received October 26, 1938.)

### 420. R. P. Dilworth: *Archimedean residuated lattices.*

Let $\Sigma$ be a noncommutative residuated lattice in which both the ascending and descending chain conditions hold. The union $m$ of all elements $x$ such that $x^\sigma=z$ for some $\sigma$ is called the *radical* of $\Sigma$. If $m=z$, $\Sigma$ is said to be *semisimple*. Elements which cover the null element $z$ are called *simple*. Let $\Sigma'$ be the lattice of elements $a$ such that $a\supset m$, and define a multiplication over $\Sigma'$ in terms of the residuals. Then $\Sigma'$ is semisimple. Let $\Sigma$ also be semisimple. Then if each element of $\Sigma$ can be represented as a union of simple elements, $\Sigma$ is a Boolean algebra. If $\Sigma$ is an arbitrary modular, semisimple lattice, it is shown that the simple elements of $\Sigma$ generate a Boolean algebra $\Sigma_B$ which is dense in $\Sigma$. Other results of a similar nature are obtained, and the structure of the lattice in the vicinity of the unit element is determined. (Received October 28, 1938.)

### 421. D. W. Hall: *On pointwise periodic homeomorphisms.*

In this note the essential features of an example by Ralph Phillips and W. L. Ayres (see a forthcoming article in Fundamenta Mathematicae) are combined with those of an example by G. E. Schweigert and the author (Duke Mathematical Journal, December, 1938) to prove the following theorem: Let $r_i$, $(i=1, 2, 3, \cdots)$, be any sequence of positive integers containing the integer 1. Then there exists in four-dimensional Euclidean space a locally connected continuum $M$ and a pointwise periodic homeomorphism $T(M)=M$ having the following properties: (a) there exists in $M$ an infinite sequence of orbits under $T$ converging to a limit set $L$ such that for every $i$ some free arc of $L$ has no interior point of period different from $r_i$; (b) the closure of every component of $M-L$ is a 2-cell. (Received October 28, 1938.)

### 422. H. J. Hamilton: *Preservation of partial limits in multiple sequence transformations.*

Consider the problems, for various pairs of classes $X$ and $Y$, of determining conditions on the matrix $\|a_{pqij}\|$ necessary and sufficient that the double sequence $\{\sigma_{pq}\}$ be of class $Y$, where $\sigma_{pq}=\sum a_{pqij}s_{ij}$, whenever the double sequence $\{s_{ij}\}$ is of class $X$, with $\lim_p \sigma_{pq}=\lim_i s_{ij}$ and $\lim_q \sigma_{pq}=\lim_j s_{ij}$ (1) for all $q$, $p$ sufficiently large, (2) for all $q$, $p$. Analogous problems are solved for sequences of dimension $n$ in general. (Received October 28, 1938.)

### 423. O. G. Harrold: *On hereditary arc sums.*

A continuous curve $M$ each of whose sub-continua is an arc sum will be called an hereditary arc sum. In this paper it is proved that this property is equivalent to each of the following: (1) each acyclic curve in $M$ is an arc sum, (2) the $M$ boundary of an arbitrary region in $M$ is countable. It is shown that every curve containing a continuum of convergence contains an acyclic non-arc sum. Thus the class of hereditary arc sums is contained in the class of hereditarily locally connected continua. (Received September 3, 1938.)

### 424. O. G. Harrold: *Concerning $2-1$ transformations on an arc.*

Let $T$ be a continuous transformation defined on the compact locally connected continuum $A$. If $B=T(A)$ is such that each point $x$ in $B$ has exactly two inverse points, $B$ is not an arc. If $A$ is an arc and $T$ is exactly $2-1$, then $B$ is not a circle. Under the same circumstances, $B$ can contain no triod composed of free arcs. Thus if $T$ is an exactly $2-1$ continuous transformation defined on an arc, the image set is not a graph. (Received September 3, 1938.)

### 425. R. D. James: *Note on integers which are not sums of three squares.*

It is well known that integers of the form $4^m(8n+7)$ are not sums of three squares. In this note it is shown that every sufficiently large integer $4^m(8n+7)$ can be written in the form $x^2+N$, where exactly one prime $p$ congruent to 3 (mod 4) divides $N$. All the other prime factors of $N$ are thus sums of two squares. In this sense the integers $4^m(8n+7)$ just fail to be representable as a sum of three squares. (Received October 28, 1938.)

426. A. D. Michal: *General Riemannian differential geometry with abstract coordinates and intrinsic inner product.*

The element of arc length in the author's previous studies on general Riemannian geometry is of the form $ds=[\delta x, g(x, \delta x)]^{1/2}$, where $[x, y]$ is the independently postulated bilinear inner product of the Banach space $E$ of coordinates. The present paper is concerned with general Riemannian spaces whose $ds=[\delta x, g(x, \delta x)]^{1/2}$ is defined in terms of the *intrinsic* "inner" product $[x, g]$ of the Banach space $E$ of coordinates. The function $[x, g]$ is bilinear on $E\bar{E}$ to the real numbers, where $\bar{E}$ is the Banach space of numerically valued linear functions on $E$. Special attention is given to those general Riemannian spaces that have a constant Riemannian curvature. (Received October 29, 1938.)

427. A. D. Michal and D. H. Hyers: *Normal coordinates in general differential geometrics with inter-space inner product.*

A theory of abstract normal coordinates in a general Michal differential geometry with a linear connection was given by the present authors in Annali della Reale Scuola Normale Superiore, Pisa, (2), vol. 7 (1938), pp. 157–176. In a memoir to appear soon in the Mathematische Annalen, the authors studied in detail the more restrictive case in which there exists an intra-space inner product in the coordinate Banach space. In the present paper the authors show how the theory remains essentially unchanged with an inter-space inner product between two Banach spaces $E$ and $E_1$. The case $E_1=\bar{E}$, the space of numerically valued linear functions, is of special interest since then the class $k^{(m)}$ of coordinate transformations is identical with the class $K^{(m)}$. (Received October 29, 1938.)

428. A. A. Shaw: *Mathematical passages in the works of David the Invincible.*

It is the purpose of this paper to give a faithful translation of all the important mathematical passages in the works of David the philosopher (the distinguished Armenian scholar of the fifth century of our era), together with a critical commentary with special reference to the related topics in Plato and Aristotle. The passages mentioned deal with: (a) theory of numbers of Pythagorean character; (b) definitions and expositions of mathematical concepts; (c) history of mathematics. The paper will also give some account of the life and works of this Armenian philosopher, generally known as David the Invincible, with illustrative photographs from the original manuscript. David occupies the same place in the history of Armenian philosophy as do Plato and Aristotle in the Greek. (Received October 18, 1938.)

429. A. A. Shaw: *An astronomical passage of antiquity.*

This paper deals with and will be a commentary on the following passage from the *Treatise on the Revolutions of the Stars*, by Bardadsan, the celebrated scientist of the second century A.D.: "Two revolutions of Saturn are equivalent to 60 years, five of Jupiter to 60 years, fourteen of Mars to 60 years, sixty of the sun to 60 years, seventy-two of Venus to 60 years, hundred fifty of Mercury to 60 years, seven hundred twenty of the moon to 60 years; and it is the same revolution for all celestial bodies, that is, the time of any one of their revolutions. From which it follows that 100 of these revolutions represent 6000 years as follows: 200 revolutions of Saturn, 500 of Jupiter, 1400 of Mars, 6000 of the sun, 7200 of Venus, 15000 of Mercury, 72000 of the moon."

Bardadsan establishes this calculation to prove that the duration of this world would be only 6000 years. (Received October 18, 1938.)

### 430. A. E. Taylor: *Reflexive Banach spaces.* I. Preliminary report.

If $E$ is a Banach space we denote by $\mathcal{E}$ the linear topological space (l.t.s.) formed from the elements of $E$ by using the weak neighborhood topology of $E$ relative to its conjugate space $E^*$. Then $\mathcal{E}^*$ denotes the corresponding space derived from $E^*$. By $\mathcal{E}_*$ we mean the l.t.s. whose elements are those of $E^*$, the topology being the weak neighborhood topology of $E^*$ relative to $E$. Then if $E_1$ and $E_2$ are two Banach spaces, and if $\mathcal{E}_1$ is linearly homeomorphic with $\mathcal{E}_{2*}$, $E_1$ and $E_2^*$ are equivalent, and $E_1$, $E_2$ are reflexive (that is, regular in the sense of Hahn). In particular, if $E_1$ and $E_2$ are the same space $E$, and if $\mathcal{E}^*$ is linearly homeomorphic with $\mathcal{E}_*$, then $E$ is reflexive. Likewise $E$ is reflexive if $\mathcal{E}$ is linearly homeomorphic with the l.t.s. $\mathcal{E}_{**}$ (the class $E^{**}$ with weak neighborhood topology relative to $E^*$). If $\mathcal{E}_1$ is linearly homeomorphic with $\mathcal{E}_2^*$ (instead of with $\mathcal{E}_{2*}$), $E_1$ and $E_2^*$ are isomorphic, but not necessarily reflexive, for it can happen that the linear mapping of $E_2^*$ on $E_1$ is not continuous as a function on $\mathcal{E}_{2*}$ to $\mathcal{E}_1$. (Received October 26, 1938.)

### 431. A. E. Taylor: *Reflexive Banach spaces.* II. Preliminary report.

Let $E$ be a Banach space, $E^*$ its conjugate space. Let $A=(\alpha)$ be a partially ordered set with the Moore-Smith property, that is, $A$ is an index set such as is used in defining Moore-Smith convergence. The space $E$ is said to be closed relative to $A$ if whenever $\{x_\alpha\}$ is a directed set (a function on $A$ to $E$) whose elements form a bounded set in $E$, there exists an element $y$ in $E$ for which $\underline{\lim}_\alpha f(x_\alpha) \leqq f(y) \leqq \overline{\lim}_\alpha f(x_\alpha)$ for each $f$ in $E^*$. Then in order that $E$ be reflexive it is necessary and sufficient that it be closed relative to every $A$. ($E$ is reflexive if every linear functional on $E^*$ is representable in the form $f(x)$, for some $x$ in $E$.) The space $E$ is said to be boundedly weakly $A$-compact if given a bounded directed set $\{x_\alpha\}$ there exists a $y$ in $E$ such that every neighborhood of $y$ (in the weak neighborhood topology of $E$) contains a cofinal subset of $\{x_\alpha\}$. Then if $E$ is boundedly weakly $A$-compact, it is closed relative to $A$. Hence if this is true for every $A$, then $E$ is reflexive. (Received October 26, 1938.)

### 432. A. E. Taylor: *Banach spaces whose unit sphere is weakly compact.*

Let $E$ be a Banach space whose unit sphere is weakly compact. Then the following things are true: (1) A necessary and sufficient condition that a series $f_1+f_2+\cdots$ of elements of the conjugate space $E^*$ be unconditionally convergent is that the series $|f_1(x)|+|f_2(x)|+\cdots$ be convergent for each $x$ in $E$. (2) A linear operation on $E$ to $l$ is completely continuous, and is defined by $\{f_i(x)\}$, where the $f_i$ form an unconditionally convergent series in $E^*$. (3) If $f_n$, $g$ are in $E^*$, $(n=1, 2, \cdots)$, and $f_n(x)\to g(x)$ for each $x$ in $E$, then $g$ is in the closed linear manifold in $E^*$ determined by $\{f_n\}$. (Received October 26, 1938.)

### 433. F. A. Valentine: *The problem of Lagrange with finite inequalities as side conditions.*

The problem treated here is a generalization of the problems of unilateral variations in the calculus of variations. The side conditions to be satisfied are taken in the form $\phi_\beta(x, y_1, \cdots, y_n)\geqq 0$, $(\beta=1, \cdots, m<n)$. Necessary conditions analogous to those of Euler, Weierstrass, Clebsch, and Mayer are obtained for a minimizing arc $E_{12}$. The paper centers around an imbedding theorem for a composite finite minimiz-

ing arc. Such an arc is one which consists of two subarcs $E_{13}$ and $E_{32}$ such that all the functions $\phi_\beta(x, y)$ are positive on $E_{13}$ except possibly at 3, whereas they are constant along $E_{32}$. This imbedding theorem enables one to establish a sufficiency theorem for the arc $E_{12}=E_{13}+E_{32}$ by strengthening the above necessary conditions in the usual way. The extension of the above results to arcs which are more general than a composite finite one is then immediate. (Received October 19, 1938.)

434. Max Wyman: *The simultaneous theory of a linear connection, and a non-holonomic linear connection in a general geometry with Banach coordinates.*

A tensor calculus for a Hausdorf space, with coordinates in a Banach space $E$, and a linear connection has been worked out. (See Michal, *General tensor analysis*, this Bulletin, vol. 43 (1937), pp. 394–401.) In this paper we consider, in addition to $E$, another Banach space $E_1$ in which non-holonomic vector fields transform according to the law $\overline{V}=M(x, V(x))$. Therefore $M(x, y)$ is a solvable linear function of $y$ on $EE_1$ to $E_1$. Non-holonomic linear connections are defined, and covariant differentials of multilinear forms are treated in the usual manner. A normal coordinate theory for $E_1$ is established, and we prove that the transformation between non-holonomic vector fields in normal coordinates is linear. Non-holonomic normal vector forms are defined, and several relations of symmetry that they satisfy are given. Some of these results are generalizations of known results for the "$n$" dimensional space. (See Michal and Botsford, Annali Di Matematica, (4), 1933–1934.) (Received October 29, 1938.)

435. Stefan Bergmann: *On the theory of harmonic functions of three variables.*

Starting from results of a previous paper (Mathematische Annalen, vol. 99 (1928), pp. 629–659, especially §3) the author studies integrals $\int[Fdx+F^*dy+F^{**}dz]$, where $F\equiv F(x, y, z)$, $F^*$, $F^{**}$ are harmonic functions and $[Fdx+F^*dy+F^{**}dz]$ is a total differential. It is shown that in certain cases there exist relations between such integrals and singularities of $F$. Also, for every $F$ there exist three harmonic functions $G\equiv G(x, y, z, X, Y, Z)$, $G^*$, $G^{**}$, depending on parameters $X$, $Y$, $Z$, such that $F(X, Y, Z)=\int[Gdx+G^*dy+G^{**}dz]$. Finally, certain properties of the representation of the space of three variables $x$, $y$, $z$ by functions $F$, $F^*$, $F^{**}$ are obtained. (Received October 11, 1938.)

436. Reinhold Baer: *Almost Hamiltonian groups.*

The elements in a group which transform every subgroup into itself form the norm of the group. The theory of the groups with cyclic norm quotient group has been developed in a previous paper. This theory establishes pretty well the relation between the norm of a given group and any element of this group with two essential exceptions: this theory does not give any information if the norm and the element in question generate together either an abelian group or a Hamiltonian group. It is the object of this note to deal with the second of these alternatives under the additional hypothesis that the norm quotient group is abelian. These apparently rather weak assumptions turn out to be very restrictive, and this makes it possible to give a fairly complete theory of this class of groups. (Received October 17, 1938.)

437. Reinhold Baer: *Groups with abelian norm quotient group.*

The norm of a group $G$ consists of all those elements in $G$ which transform every subgroup of $G$ into itself. It is the object of this note to describe the group of automorphisms which are induced in the norm of $G$ by the elements of $G$, provided the norm quotient group is abelian. (Received October 17, 1938.)

### 438. Reinhold Baer: *Nets and groups.*

The combinatorial properties underlying the configuration of three pencils of parallel straight lines in the plane have found their condensation in the concept of *net*. It is the object of this paper to establish the equivalence of the theory of nets with a certain chapter in the theory of groups. (Received October 17, 1938.)

### 439. M. A. Basoco: *On certain arithmetical functions due to G. Humbert.*

G. Humbert has pointed out (Comptes Rendus, vol. 158 (1914), pp. 220, 294) the existence of a class of entire functions having interesting arithmetical properties. In this paper, Humbert's note is extended and supplemented; a set of functional equations is obtained which serve to define Humbert's functions uniquely. Advantage is taken of the fact that these functions are special cases of certain others which the present author obtained in a former paper (American Journal of Mathematics, vol. 54 (1932), pp. 242–252) to relate them to the theta-functions by means of certain identities. The paraphrases of these are related to the representation of a number as the sum of five squares. Some applications involving the greatest integer function are also given. (Received October 18, 1938.)

### 440. O. E. Brown: *A least squares machine.* Preliminary report.

This paper is a description of a proposed device for the solution, by mechanical means, of a system of linear algebraic equations. It gives approximate solutions, which, by an easy process of iteration, approach the exact values. While the machine may be employed for any system whatever, within its capacity, it is especially valuable when the system is inconsistent, as is usually the case when the constants of the system are obtained by observation. In this case identical springs, one for each equation, introduce modifications in the constant terms and allow the modified system to be solved. The extent and direction of the distortions of the springs determine the residuals. If Hooke's law is assumed to hold for each spring, it follows that the solution of the modified system satisfies the corresponding system of normal equations and, therefore, the sum of the squares of the residuals is a minimum. (Received October 28, 1938.)

### 441. R. S. Burington: *On the congruence of matrices and associated circavariant matrices.*

Let $A$, $B$, $C$, and $P$ be matrices with elements over a field. Let $C^{s_1 \cdots s_t}_{r_1 \cdots r_t}$ denote the matrix obtained from $C$ by deleting rows $r_1 \cdots r_t$ and columns $s_1 \cdots s_t$. If (1) $B = P'AP$, where $P$ is nonsingular, and if (2) $B^{s_1 \cdots s_t}_{r_1 \cdots r_t} = P'^{r_1 \cdots r_t}_{r_1 \cdots r_t} A^{s_1 \cdots s_t}_{r_1 \cdots r_t} P^{s_1 \cdots s_t}_{r_1 \cdots r_t}$, then $A^{s_1 \cdots s_t}_{r_1 \cdots r_t}$ is called a *circavariant matrix* of $A$ under the congruence (1). In this paper a theory of circavariant matrices is initiated. General theorems relating to the restrictions that must be imposed on $P$ in order that one or more of the set $A^1, A^2, \cdots, A^{s_1 \cdots s_t}_{r_1 \cdots r_t}, \cdots$ be circavariant are given. Theorems on the congruence of matrices with $P$ in a modified $m$-affine space are obtained, together with a set of canonical forms. The relation to the author's previous paper (Transactions of this Society, vol. 38 (1935), pp. 163–176) is discussed. The results of this paper are fundamental to the construction of a general theory of relative-equivalent electrical networks, to be considered later. (Received October 25, 1938.)

442. Max Coral: *On fields for double integral variation problems.*

The form of the most general field for a multiple integral variation problem has been determined recently by Bliss, Smiley, and Alaoglu. In the case of a double integral, the invariant integral of the field involves a quadratic form in the derivatives of the functions along which the integral is computed. The complete determination of the field demands the determination of the coefficients of this quadratic form, concerning which it is known only that they satisfy a certain system of partial differential equations. In the present paper, the author completes the theory as follows: if the extremals of the field are known, the coefficients of the quadratic form can be determined from them without integration of the partial differential equations, and they are unique up to additive functions which remain constant on each extremal. Also, if a region $F$ is simply covered by a family of extremals, conditions on the family are found which are necessary and sufficient in order that $F$ form a field. (Received October 26, 1938.)

443. H. V. Craig: *On extensors and a euclidean basis for higher order spaces.*

This paper is a continuation of the paper, *On tensors relative to the extended point transformation*, American Journal of Mathematics, vol. 59 (1937). The writer considers arbitrary curves $x^a = x^a(t)$ in an $x$-space of $N$ dimensions and sets up a correspondence with a euclidean space of $L$ dimensions with rectangular cartesian coordinates $y^i$. The correspondence is obtained by equations expressing the $y$'s as functions of the $x$'s and their derivatives with respect to $t$ up to order $M$. If $M$ is zero, a Riemannian subspace is obtained. The geometry of the case $M>0$ is studied, and it is found that extensors play a role somewhat similar to that of tensors in Riemannian geometry. A metric extensor is obtained and from it "Christoffel symbols." Some of the Christoffel symbols of reduced range are tensors, others extensors. Excovariant derivatives (analogous to covariant derivatives) are formed and investigated. Generalizations of certain identities of Finsler geometry are found. (Received October 14, 1938.)

444. J. J. DeCicco: *The analogue of the nine-point circle in the associated Kasner plane.*

This is a continuation of two papers by Kasner (Science, vol. 85 (1937), pp. 480–482, and Proceedings of the National Academy of Sciences, vol. 23 (1937), pp. 337–341) and abstract 42-11-398 by Kasner. A *simple horn-set* is the totality of all curves (third order elements) which possess a common point and direction. Let $x=\gamma$, $y=d\gamma/ds$, where $\gamma$ is the curvature and $s$ is the arc length. A simple horn-set is then called the *associated Kasner plane* $K_2$, where any point of $K_2$ is a curve $(x, y)$ of the simple horn-set. The group of conformal transformations induces in $K_2$ the special affine three-parameter group $G_3$: $X=mx+h$, $Y=m^2y+k$. In this paper, the triangle geometry of the group $G_3$ is studied. Many theorems of this geometry are completely analogous to those of euclidean geometry, but the majority are entirely different. The analogs of the circumscribed, inscribed, and nine-point circles are obtained and entirely new relationships between them are discovered. The analogs of the Simpson line, Menelaus' and Ceva's theorems, and the Brocard points are also obtained. (Received October 24, 1938.)

445. J. M. Dobbie: *A note on a convergence proof for Fourier series.*

Stone (Transactions of this Society, vol. 28 (1926), pp. 695–761) proved that the expansion problem associated with the differential system $u^{(n)}+\lambda u=0$, $u^{(r)}(0)-u^{(r)}(1)=0$, $(r=0, 1, 2, \cdots, n-1)$, in which the boundary conditions are regular, gives rise to Fourier series. In this note it is shown how the convergence proof for such a system can be made to depend upon the contour integration proof for the first order problem given by Birkhoff (Transactions of this Society, vol. 9 (1908), pp. 373–395), an extension of which was made to the second order problem by Carman (this Bulletin, vol. 30 (1924), pp. 410–416). (Received October 26, 1938.)

446. Wallace Givens: *Signatures of Lorentz matrices.*

It is first proved that every Lorentz matrix of order $n$ can be expressed as the product of involutory Lorentz matrices having the characteristic equation $(x+1)(x-1)^{n-1}=0$. Such an involution $I$ is called spatial or temporal according as $Q(y)>0$ or $<0$, where $y$ is a solution of the equations $Iy=-y$ and $Q(x)$ is the invariant quadratic form. Two products of such involutions cannot be equal unless the number of spatial (temporal) involutions in one product is congruent (mod 2) to the number of spatial (temporal) involutions in the other. This gives an algebraic treatment of the signatures of Lorentz matrices which seems more direct than that employing the spin representation. (Received October 28, 1938.)

447. D. K. Kazarinoff: *Remark concerning Desargues' triangle theorem.*

This paper gives a projective generalization of a theorem due to Möbius: "If the plane of one of two triangles in perspective is fixed while the plane of the other rotates about the axis of perspective, then the center of perspective describes a circle in a plane perpendicular to the axis of perspective and the center of the circle lies in the fixed plane." Let $A_1B_1C_1$ be a fixed triangle and $A$, $B$, $C$ be any three collinear fixed points chosen, respectively, on $B_1C_1$, $C_1A_1$, $A_1B_1$. Consider any two figures $(A_2)$ and $(B_2)$ homological to each other with the center of homology $C$ and the arbitrary chosen plane of homology $\pi$, and a third figure $(C_2)$ homological to $(A_2)$ and $(B_2)$ with the centers $B$ and $A$, respectively, and the plane $\pi$. Our theorem is: If a variable triangle $A_2B_2C_2$ deforms in such a way as to have its vertices as the corresponding points of $(A_2)$, $(B_2)$, $(C_2)$, then $A_2B_2C_2$ remains always in perspective to $A_1B_1C_1$ and the center of perspective $D$ describes a figure $(D)$ which is homological to $(A_2)$, $(B_2)$, $(C_2)$ with the centers of homology $A_1$, $B_1$, $C_1$, respectively, and the plane of homology $\pi$. (Received October 28, 1938.)

448. J. F. Kenney: *The regression systems of two sums having random elements in common.*

Let $f(x)$ be the probability function of $x$, and suppose the mean $\int_{-\infty}^{\infty}xf(x)dx$ exists. A sample of $n$ independent variates is taken from the population represented by $f(x)$ and the sum $y=\sum_1^n x_i$ is formed. From this sample $k<n$ values are chosen at random, and a sample of $m-k$, $(m\leqq n)$, independent variates $x_i'$ is taken from the population $f(x)$. The sum $z=\sum^k x_i+\sum_{k+1}^m x_i'$ is then formed. The problem of determining the regression systems of $y$ on $z$ and $z$ on $y$ in the population resulting from repeated samples is discussed elsewhere in the literature by Fischer and others. In the present paper a brief and elegant solution is given by means of characteristic functions. (Received October 28, 1938.)

449. Cornelius Lanczos: *A simple interpolation method for the representation of rugged curves.*

If a function is given in equidistant ordinates, the customary method of interpolation makes use of a difference table. This method fails if an empirical function is so unsmooth ("rugged") that the successive differences show no regularity and do not converge toward zero. Such a function may still be developable into a sufficiently convergent Fourier series, and a simple interpolation formula then holds. The validity of the method presumes that the Fourier series is practically limited to a number of terms not exceeding the number of points in which the function is given. In order to satisfy this condition only functions are considered which vanish at the two end points of the given range. The range is chosen as the half period and only sine terms are used, which permits continuity of function and first derivative and thus provides sufficient convergence. If the given values do not satisfy those boundary conditions, we define $\phi(x)=f(x)-(\alpha+\beta x)$, $\alpha$ and $\beta$ suitably chosen, and interpolate $\phi(x)$. This gives satisfactory results for empirical curves which are not smooth enough for interpolation by differences and yet not too rugged to prohibit any reasonable interpolation. (Received October 24, 1938.)

450. Saunders MacLane: *Steinitz field towers for modular fields.*

If a field $K$ of characteristic $p$ has a transcendence basis $T$ over a perfect subfield $P\subset K$, then $T$ may be called a separating transcendence basis if every element of $K$ is separable and algebraic over $P(T)$. In an investigation of the structure of complete fields with valuations, Hasse and Schmidt have used, but not proved, a theorem to the effect that, in any imperfect field $K$, one can select a suitable subfield $L$ such that $K$ can be approximated by a tower of fields $S_n$ where $S_n$ has a separating transcendence basis over $L$. Furthermore, they assert conditions which imply that $S_n = L(S_{n+1}^p)$. The present paper offers examples disproving the existence of such approximating towers in general, but provides a construction for modified such towers in a large class of cases. (Received October 18, 1938.)

451. J. R. Musselman: *The equation of motion of equal maps.*

The theory of rotation enables us to move an object from one position on a plane to another position on the same plane. The purpose of this paper is to discuss a type of motion which will send an object into three or more positions consecutively on the same plane. Or, if we prefer, we may think of three or more equal objects (for convenience called maps) lying on a plane and ask for the equation of motion which will pick up all of them. The relationships existing among corresponding points and among corresponding lines in the maps are pointed out. (Received October 30, 1938.)

452. C. J. Nesbitt: *Note on primary algebras.*

Let $A$ denote an algebra of finite rank over a field $K$, and let $N$ be its radical; $A$ is called primary if it possesses a unit element and the quotient algebra $A/N$ is simple. Methods in the theory of regular representations of algebras (see Brauer and Nesbitt, Proceedings of the National Academy of Sciences, vol. 23) are here applied to a primary algebra $A$ over an algebraically closed field $K$ to obtain a reduced form for the regular representation of $A$. By means of this reduced form primary algebras which are also symmetric (see the above reference and Nakayama and Nesbitt, Annals of Mathematics, (2), vol. 39 (1938), p. 659) are studied and, in particular, the linear functions $\Phi$ of the elements of $A$ with the property $\Phi(\alpha\beta)=\Phi(\beta\alpha)$, for any

$\alpha$, $\beta$ belonging to $A$, are discussed. The results have bearing on the modular representations of finite groups. (Received October 28, 1938.)

### 453. Rufus Oldenburger: *Higher-dimensional determinants.*

It has not been known that higher dimensional determinants arise in simple ways in the theory of ordinary determinants. In this paper we show that the coefficients in the expansion of an ordinary determinant whose elements are linear forms are three-dimensional determinants. Also, the derivative of a determinant whose elements are differentiable functions of $x$ with respect to $x$ is a three-dimensional determinant. In analogous ways $(p+1)$-dimensional determinants arise in the study of $p$-dimensional determinants. From these considerations it follows that invariant ranks of forms defined elsewhere by the author in terms of the vanishing and nonvanishing of $p$-way determinants may be defined in terms of maximum ranks of $(p-1)$-way generalized Hessian matrices. (Received October 26, 1938.)

### 454. A. E. Pitcher: *A simpler proof of a theorem of Morse.*

Morse, Transactions of this Society, vol. 27 (1925), and Morse and Van Schaack, Annals of Mathematics, vol. 35 (1934), obtain a complete set of relations between the numbers of critical points of various types and the connectivities (mod 2) of the region in which the function is defined. The function is required to be of class $C^2$, to have only nondegenerate critical points, and to satisfy simple boundary conditions. A principal intermediate theorem, proved completely only in the first paper, is the following: If the function $f$ has exactly one critical point at the level $c$ and $k$ is its type, and if $e$ is a sufficiently small positive number, then the differences $\Delta R_i$ in the connectivities of the sets $f \leqq c-e$ and $f \leqq c+e$ are all 0 except that $\Delta R_k = 1$ or $\Delta R_{k-1} = -1$. The two cases are mutually exclusive. The proof given by Morse is greatly simplified by the author through the use of the Mayer-Vietoris formulas, which appeared several years after the first paper by Morse. Formulas equivalent to the Mayer-Vietoris formulas were used by A. B. Brown in a study of isolated critical points, possibly degenerate, of an analytic function. (Received October 28, 1938.)

### 455. E. D. Rainville: *Linear differential invariants related to the Laplace integral transformation.*

Certain linear differential equations are left invariant when subjected to the Laplace integral transformation. Such invariance depends essentially on two properties of the Laplace operator. This suggests the introduction of a linear operator $\sigma$ which has the above mentioned properties in common with the Laplace operator and which is defined with reference to certain linear differential expressions. Let a linear differential form be a finite expression linear and homogeneous in a function and its derivatives, with coefficients which are polynomials in the independent variable. We study the properties of $\sigma$ when applied to such forms. In four ways all forms which are invariant or pseudo-invariant under $\sigma$ are classified. The general operational equation in $\sigma$ with constant coefficients is solved in terms of linear differential invariant forms. Some of the results are extended to the case of forms in two independent variables subjected to iterated $\sigma$ transformations. A generalization of $\sigma$, in which the generalized operator is applied to expressions of which linear differential forms are a very special case, is studied with particular reference to invariant expressions. (Received October 27, 1938.)

### 456. E. H. Rothe: *Asymptotic solution of a boundary value problem.*

Let $y$ be the solution of the boundary value problem $y''+\mu(y'-y)=\mu f(x)$, $y(0)=y(1)=0$. It is shown that for $\mu\to+\infty$, $y$ approaches that solution of $z'-z=f(x)$ which is zero for $x=1$. In the case of $y''-\mu(y'+y)=\mu f(x)$, $y$ approaches that solution of $-z'-z=f(x)$ which is zero for $x=0$. (Received October 28, 1938.)

### 457. Leonard Bristow: *Expansion of functions in solutions of functional equations.*

The functional operator $L(x, \lambda)$ is assumed to have the following property: $L(x, \lambda)\to x^p = x^p f(x, \lambda, p) = x^p \sum_\mu f_\mu(p, \lambda) x^\mu$, the series converging suitably for all values of $p$ and the parameter $\lambda$. Conditions upon the function $f(x, \lambda, p)$ are found so that there exists a set of values $\{\lambda_m\}$, $(m=0, 1, 2, \cdots)$, such that the equation $L(x, \lambda)\to y(x)=0$ has a set of solutions $\{y_m(x)\}$, $y_m(x)=x^m\sum_s \alpha_{s,m}x^s$, which can be used to expand an arbitrary analytic function $f(x)$; that is, $f(x)=\sum_m y_m(x)$. The expansion converges and represents the function in a circle about the origin. Special cases include: (a) linear differential equations with a regular singular point, (b) Volterra homogeneous integral equations with a regular singularity, (c) linear $q$-difference equations. (Received September 28, 1938.)

### 458. Jesse Douglas: *Minimal surfaces of higher topological structure.* II.

This is a continuation, with more details, of the author's papers on the same subject recently presented to the Society; see this Bulletin, vol. 44 (1938), pp. 487–488. (Received October 3, 1938.)

### 459. Karl Menger: *On non-euclidean and affine geometry.*

The whole geometry of Bolyai and Lobatchewski, including the chapters on parallelism, order, congruence, and convergence, can be based on assumptions concerning the operations of joining and intersecting. The whole affine geometry can be obtained by introducing ideal elements into the non-euclidean geometry. (The results of this paper are published in the Proceedings of the National Academy of Science, October, 1938.) (Received October 3, 1938.)

### 460. G. H. Peebles: *Some generalizations of the theory of orthogonal polynomials.*

If real constants $a$, $b$, $c$, $d$, and $e$ are chosen within certain ranges, and if $\rho(x)$ is a non-identically vanishing solution of the differential equation $\rho'(x)/\rho(x)=N(x)/D(x)$, $N(x)=ax+b$, $D(x)=cx^2+dx+e$, the polynomials $q_0(x)=1$, $q_n(x)=(1/\rho)d^n(\rho D^n)/dx^n$, $(n=1, 2, 3, \cdots)$, are an orthogonal set with respect to $\rho(x)$ over an interval dependent on $c$, $d$, and $e$. For values outside these ranges, the set of polynomials loses the property of orthogonality. The purpose of the first part of this paper is to show that the non-orthogonal sets possess properties ordinarily associated with orthogonal sets, namely, each set satisfies the general recursion formula of orthogonal polynomials, each set of a restricted class satisfies the Christoffel-Darboux identity, and each set of a further restricted class has the property of representing suitable functions by means of convergent series. The second part gives a method of constructing orthogonal polynomials for a class of weight functions which change sign in the interval of orthogonality. The method used leads to theorems on the vanishing of determinants formed from the polynomials orthogonal with respect to weight functions which are positive in the interval of orthogonality. (Received October 21, 1938.)

### 461. M. S. Robertson: *On certain power series having infinitely many zero coefficients.*

Let $f(z)=z+\sum_2^\infty a_nz^n$ be regular for $|z|<1$, and within this circle take on real values for, and only for, real values of $z$. The author shows that if $a_n=0$ for $n\equiv 0$ (mod $p$), where $p$ is any odd integer greater than one, then the coefficients $a_n$ are uniformly bounded: $|a_n|\leqq p-1$. This theorem is not true when $p$ is an even integer. For $p=3$ the sharp inequality $|a_n|\leqq$max. on $(0\leqq\phi\leqq(\pi/3))$ of $|\cos n\phi/\cos\phi|<2$ is obtained. In this inequality the equal signs are seen to hold for the function $(z-z^2\sec\phi+z^3)\cdot\{1-2z\cos(\pi/3+\phi)+z^2\}^{-1}\cdot\{1-2z\cos(\pi/3-\phi)+z^2\}^{-1}$, where $\phi$ is chosen as that value for which $|\cos n\phi/\cos\phi|$ attains its maximum over $(0,\pi/3)$. This extremal function varies with $n$. The rate of growth of $f(z)$ is given by $|f(re^{i\theta})|\leqq 4r/(3^{1/2}(1-r^2))$. These results apply a fortiori to functions $f(z)$ univalent within the unit circle and real on the real axis. (Received September 29, 1938.)

### 462. S. M. Ulam: *On the distribution of a general measure in any complete metric separable space.*

Let $\mu(A)$ be a completely additive measure defined for a class of subsets of a complete metric separable space $E$ such that open subsets are measurable. There exist compact subsets of $E$ having positive measure and a subset representable as the sum of countably many compact sets having a measure equal to that of the space $E$. (Received October 4, 1938.)

### 463. V. W. Adkisson: *Closed disconnected sets with unique maps in the plane.*

An $E$-set is a closed bounded and disconnected plane point set, each component of which is locally connected, not more than a finite number being of diameter greater than any positive number. The following theorem is proved: An $E$-set $M$ has a unique map in the plane if and only if one of the following sets of conditions holds: (1) each component of $M$ is an arc or a point, and no interior point of an arc $t$ is a limit point of $M-t$ (referred to as an M.K. set); (2) one component $N$ of $M$ is a triod, and $M-N$ plus the three end points of $N$ is an M.K. set; (3) one component $N$ of $M$ contains a closed 2-cell $C$ and $N-C$ consists of at most a countable number of arcs $a_1, a_2, \cdots$, such that $\bar{a}_i\cdot\bar{a}_j=0$, $(i\neq j)$, and each $\bar{a}_i\cdot C$ is a single point which lies on the boundary of $C$; each component of $M-N$ is an arc or point such that no interior point of an arc $t$ (of $M-N$) is a limit point of $M-t$; the points of $(\overline{M-N})\cdot N$ are noncut points of $N$. (Received October 24, 1938.)

### 464. Marguerite D. Darkow: *Generalized center-circles.*

The method used by Morley to obtain his chain of theorems on center-circles is capable of generalization and yields infinitely many such chains. Associated with a set of $n$ mutually intersecting lines, there are center-circles of order $1, 2, \cdots, n-2$. Concerning these, the following theorems hold: (1) The $i$th order center-circle of an $n$-line passes through the centers of the $n$ $i$th order center-circles of the $n$ sets of $(n-1)$ lines contained in the given $n$-line. (2) The $n$ $i$th order center-circles of the $n$ sets of $(n-1)$ lines formed from a given $n$-line meet in a point which, when the $i$th order center-circle of the $n$-line is not a point circle, is the node or focus of the limaçon (or the center of a circle, that is, a degenerate limaçon) to which the $n$ given circles are tangent. In particular, when $i=(n-2)/2$, this point of intersection lies on the $i$th order center-circle of the $n$-line. (Received October 18, 1938.)

465. Jesse Douglas: *The analytic prolongation of a minimal surface over a rectilinear segment of its boundary.*

Schwarz proved the theorem that if a minimal surface contains a straight line in its interior, then this straight line must be an axis of symmetry of the surface. All proofs so far given of this theorem make essential use of the *interior* position of the line with respect to the minimal surface. For certain interesting applications, however, it is important to have stronger information. Suppose the straight line segment $l$ is part of the *boundary* of a portion $M$ of a minimal surface. Let $M$ be rotated about $l$ through 180°, producing $M'$. Is then $M'$ the analytic prolongation of $M$ across $l$? The present paper proves the affirmative answer. An application is then given to the construction of a minimal surface bounded by two given interlacing circles; thus a general theorem is illustrated concerning interlacing contours given by the author in the Journal of Mathematics and Physics, vol. 10 (1931), p. 351. This paper will appear in the Duke Mathematical Journal. (Received October 27, 1938.)

466. M. H. Heins: *Interpolation to functions harmonic and positive or bounded in the unit circle.*

Let $P_h$ denote the class of functions $u(z)$ harmonic and positive for $|z|<1$. Suppose $|z_j|<1$, $(j=1, 2)$, and let $D(z_1, z_2)$ denote the hyperbolic distance between $z_1$ and $z_2$. Then $u(z_2)e^{-2D(z_1,z_2)}\leqq u(z_1)\leqq u(z_2)e^{2D(z_1,z_2)}$ for $u\subset P_h$. This inequality enables one to study the general problem of the existence of a function $u(z)\subset P_h$ for which $z_k$, $(|z_k|<1)$, and $u(z_k)=u_k>0$ are preassigned, $(k=1, 2, \cdots, n$ or $k=1, 2, \cdots)$. The inequalities (1) $u_je^{-2D(z_j,z_k)}\leqq u_k\leqq u_je^{2D(z_j,z_k)}$, $(j\neq k; j, k=1, 2, \cdots, n$ or $j, k=1, 2, \cdots)$, constitute a necessary condition for the existence of such a $u(z)$. Conversely, if (1) is satisfied, either ($1^0$) for some $j^*$, $k^*$ one of the weak inequalities (1) is an equality and there exists a unique function $u(z)\subset P_n$ for which $u(z_{j*})=u_{j*}$, $u(z_{k*})=u_{k*}$, and a necessary and sufficient condition that the problem be compatible is that the remaining $u_k$ be equal to $u(z_k)$; or ($2^0$) all the inequalities (1) are satisfied strongly and there exists a function $u(z)\subset P_h$ with the desired interpolation properties, and all such functions can be determined. The interpolation problem for the class of functions $u(z)$ harmonic for $|z|<1$, $-1<u(z)<1$, $|z|<1$, can be treated similarly. (Received October 31, 1938.)

467. F. B. Jones: *Concerning simple linear Moore spaces and simple continuous curves.*

It is first established that if the author's Axiom $5_1$* (Transactions of this Society, vol. 42 (1937), p. 54) holds true in a Moore space, the space is locally connected, if connected. A simple linear space is one in which the following axiom holds true: If $P$ is a point of a region $R$, there exists in $R$ a domain $D$ containing $P$ such that $D$ has at most two boundary points. The main result of the paper is that every nondegenerate component of a simple linear Moore space is a simple continuous curve. (A Moore space is a space in which Axiom 0 and parts 1, 2, and 3 of Axiom 1 of R. L. Moore's *Foundations of Point Set Theory* hold true.) (Received October 25, 1938.)

## 468. F. B. Jones: *Concerning the boundary of a complementary domain of a continuous curve.*

Suppose that a space satisfies Axioms 0–4 of R. L. Moore's *Foundations of Point Set Theory*. Among others, the following results are established. I. If $K$ is a locally compact continuum lying in the boundary of a connected domain $D$, then in order that $K$ be a continuous curve it is necessary and sufficient that $K$ be a subset of a continuous curve which contains no point of $D$. II. If $D$ is a complementary domain of a locally compact continuous curve $M$, and $E$ is a point of $S-\overline{D}$, then every nondegenerate component of the outer boundary of $D$ with respect to $E$ is a simple continuous curve. III. Under the hypothesis of II, the outer boundary of $D$ with respect to $E$ is either acyclic or a simple closed curve. IV. If $K$ is the boundary of a complementary domain $D$ of a locally compact continuous curve $M$, $\beta$ is the outer boundary of $D$ with respect to a point $E$ of $S-\overline{D}$, and $H$ is a component of $K-\beta$, then $\overline{H}$ is a continuous curve having only one point in $\beta$. (Received October 25, 1938.)

## 469. J. L. Kelley: *Fixed sets under homeomorphisms.*

The following generalization of a theorem by W. L. Ayres (*Some generalizations of the Scherer fixed point theorem*, Fundamenta Mathematicae, vol. 24 (1925), pp. 125–130) concerning homeomorphisms of locally connected continua into themselves is proved: If $T$ is a homeomorphism carrying a compact continuum $M$ into a subset of itself, there exists a continuum $H\subset M$ without cut points of itself and with $T(H)=H$. If $M$ is decomposed into continua $E_0$ maximal (see G. T. Whyburn, *Concerning maximal sets*, this Bulletin, vol. 40 (1934), pp. 159–160; also, *Cyclic elements of higher order*, American Journal of Mathematics, vol. 56 (1934), pp. 133–146) with respect to the property of containing no cut points, we may state this theorem: If every homeomorphism $T$ carrying any set $E_0\subset M$ into a subset of itself leaves a fixed point, then every homeomorphism of $M$ into itself leaves a fixed point; that is, the fixed-point property for homeomorphisms is $E_0$ extensible. (Received October 21, 1938.)

## 470. Walter Leighton: *New convergence theorems for continued fractions.*

This paper provides an infinite sequence of convergence theorems, independent of earlier known theorems, for continued fractions of the form (1): $1+K_1^\infty(a_n/1)$, where the $a_n$ are complex quantities not equal to zero. The results are combined into the following principal theorem: If the denominators $B_r$ of the first $k$ approximants of the continued fraction (1) are not zero, if the numbers $B_{k-1,\lambda}$ are not zero, if $|a_{nk+s+1}B_{k-2,nk+s+1}|\leqq m<1$, for each value of $s$ except those congruent to $r_0 (\mathrm{mod}\ k)$, and if $|B_rB_{k+r}|\geqq|B_r|+|a_1a_2\cdots a_{r+1}B_{k-1,r+1}|$, $|B_{2k-1,\lambda}|\geqq|B_{k-1,\lambda}|+|a_{\lambda+1}a_{\lambda+2}\cdots a_{\lambda+k}B_{k-1,r+k}|$, $(r=0, 1, 2, \cdots, k-1; \lambda=1, 2, 3, \cdots)$, the continued fraction converges. The number $r_0$ is any fixed integer. The notation is that of Perron. Each value of $k=2, 3, 4, \cdots$ yields a theorem for each value of $r_0=0, 1, 2, \cdots, k-1$. (Received October 31, 1938.)

## 471. Norman Levinson: *General gap Tauberian theorems.* II.

Continuing the problem begun in the Proceedings of the London Mathematical Society, it is shown that the hypothesis for the general higher indices theorem given there is essentially the best possible. It turns out that such kernels as $K(x)=e^{-x^2/2}$ require larger gaps than the higher indices type in order that an analogous theorem be true. For example, instead of $\lambda_{n+1}-\lambda_n \geqq D>0$ as in the general higher indices theorem, for $e^{-x^2/2}$ there is needed $\sum 1/\lambda_n < \infty$. The general result (stated very roughly) is that if $\lim_{x\to\infty}\sum_0^\infty a_n \int_{\lambda_n - x}^\infty K(y)dy = s$, if the Fourier transform of $K(x)$ is $k(u)$, if $\Lambda(x)$ is the number of $\lambda_n < x$, if $\alpha(u)$ is monotone increasing and $|k'(u)/k(u)| \leqq \alpha(u)$, and if $\int_1^\infty du/u^2 \int_1^{\alpha(u)} \Lambda(x)dx/x < \infty$, then $\sum a_n = s$. Although it is known that there is no unrestricted Tauberian theorem for $e^{-x^2/2}$ if $\{\lambda_n\}$ satisfies nothing more than $\lambda_{n+1}-\lambda_n \geqq D>0$, there is a very mildly restricted result in this case which, stated roughly, requires that $a_n = O(e^{n^2/2 - cn \log n})$. Here again there is a general theorem with the same range of application as the theorem stated above. Examples can be given to show that both general theorems referred to here are the best possible. (Received October 27, 1938.)

## 472. Norman Levinson: *On functions of minimal exponential type.*

Pólya once stated the following theorem for which many proofs have been given: Let $F(z)$ be an entire function of minimal exponential type; if $F(\pm n)$ is uniformly bounded, then $F(z)$ is a constant. The author proved (American Journal of Mathematics) that it suffices if $F(\lambda_n) = O(1)$, $(-\infty < n < \infty)$, if $|\lambda_n - n| < M$. This result is now extended to the theorem: The function $F(z)$ is constant if $F(\lambda_n) = O(1)$ and $|\lambda_n - n| < |n|/(\log|n|)^{1+\epsilon}$, $(-\infty < n < \infty)$. (This last inequality can be made more precise by stating it as a test depending on the convergence of an integral.) What is quite astounding is that this result is the best possible, in the sense that an example can be given of an entire function of minimal type bounded for $z = \lambda_n$ where $|\lambda_n - n| < |n|/\log |n|$, $(-\infty < n < \infty)$, which is not a constant. This result deviates very much from existing density theorems in that it goes as far as it does and yet it need not be true for a set $\{\lambda_n\}$ having a density $D>0$. (Received October 27, 1938.)

## 473. O. F. G. Schilling: *A remark on normal extensions.*

The author investigates the theory of normal extensions $K$ over function fields of one variable $k$. The groups $G$ to be considered are groups of order $l^m$, $l$ a prime. It turns out that, except for special cases, one can reestablish Riemann's existence theorem by purely algebraic methods. The case that $l$ is equal to the characteristic of the underlying field $k$ requires a different treatment. It is found that by prescribing a single branch point, all groups $G$ of order $l^m$ can be realized as Galois groups. (Received October 13, 1938.)

## 474. A. R. Schweitzer: *On the construction of "configurational" sets of ordered dyads in the foundations of geometry.*

Analogous to his construction of a complete set of open linear chains of dyads in Chapter 2 of his theory of geometrical relations (American Journal of Mathematics, vol. 31), the author constructs complete systems of sets of ordered dyads in $n$ elements ($n = 1, 2, 3, \cdots$) on the basis of the self-conjugate dyad $\alpha\alpha$ by means of two principles of generation: (1) successive adjunction of self-conjugate dyads, and (2) successive replacement of any dyad by connected dyads of which it is the resultant. These sets of dyads are placed in two categories: (1) primarily with, and (2) without reference to order. Sets in the second category are abstractly identical with substitutions, and their group properties are derived entirely within the frame of development of Chapter 2 of the author's theory cited. Graphically, the latter sets of dyads are represented by "constellations" or "configurations," that is, combinations of point digons ($\alpha\alpha$), oriented digons ($\alpha\beta$, $\beta\alpha$), oriented trigons ($\alpha\gamma$, $\gamma\beta$, $\beta\alpha$), and so on. Transition to the corresponding representation of abstract groups is made. The author refers to Chapter 3 of his theory (ibid., vol. 31), to mathematico-psychological theories of "form" of A. B. Kempe and K. Koffka and articles by A. Cayley (ibid., vols. 1, 11), E. H. Moore (ibid., vol. 18), M. Dehn (Mathematische Annalen, vol. 71), and the *Habilitationsschrift* of W. Threlfall (1932). (Received October 24, 1938.)

## 475. W. J. Trjitzinsky: *Some general developments in the theory of functions of a complex variable.*

The present paper (to appear in Acta Mathematica), and another by the author (*Théorie des fonctions d'une variable complexe définies sur des ensembles généraux*, Annales Scientifiques de l'École Normale, 1938, pp. 119–191), contain an extensive investigation of various classes of functions of a complex variable $f(z)$, each possessing either a unique derivative over some measurable set $E$ or essentially equivalent properties. Also functions $f(z)$ are considered which are limits of sequences of analytic functions $f_\nu(z)$, where each $f_\nu(z)$ is defined over $O_\nu \supset E$. The theory includes as special subcases the theories of analytic, quasi-analytic, and other significant classes of functions. It is important that their representation involves, essentially, integrals over two-dimensional sets of $(\zeta - z)^{-1} d\mu$ where $\zeta = \xi + i\eta$ is the variable of integration and $\mu$ is an additive function of Lebesgue measurable sets (absolutely continuous or not). This affords an effective analytic study of the functions in question. Various degrees of "rarefication" of $\mu$ in the vicinity of a suitable set $H \subset E$ will secure various preassigned regularity properties; for example, representation as limits of rational functions, differentiability, assigned "degree" of continuity, and unique determination by values on an arc, or on a set of linear positive measure, or by values of derivatives of all orders at a point or on a nondense set, and so on. (Received October 27, 1938.)