CONCERNING CONTINUA IN A SEPARABLE SPACE WHICH DO NOT CROSS*

O. H. HAMILTON

In working with collections of continua it is sometimes useful to know something of the character of the point set consisting of all the points common to two or more members of the collection. Also it is of advantage to know conditions under which we may subtract a countable number of continua from the collection and have left a collection of mutually exclusive continua. The theorem which we shall prove may aid in answering questions of this nature.

All definitions and discussions will refer to point sets in a connected and locally connected separable space S.

DEFINITION 1. *If g_1 and g_2 are two continua each of which separates S, g_1 will be said to cross g_2 provided there exist two complementary domains of g_2 (maximum connected domains of $S - g_2$) which contain points of g_1.*

DEFINITION 2. *If (1) M_1, M_2, and M_3 are three point sets in S, (2) g_1 and g_2 are two continua in S, (3) M_1 is in a complementary domain D_1 of g_1 which does not contain a point of g_2, (4) M_2 is in a complementary domain D_2 of g_2 which does not contain a point of g_1, and (5) M_3 is in a complementary domain D_3 of $g_1 + g_2$ distinct from D_1 or D_2, then $g_1 + g_2$ will be said to separate M_1, M_2, and M_3 symmetrically with respect to M .*

LEMMA 1. *If (1) g_1 and g_2 are two continua each of which separates S, (2) each separates some complementary domain of the other, and (3) neither crosses the other, then there exist domains D_1, D_2, and D_3 such that $g_1 + g_2$ separates D_1, D_2, and D_3 symmetrically with respect to D_3.*

PROOF. $S - g_1$ is the sum of two mutually separated point sets S_1 and S_2. One of these, say S_1, is such that the continuum g_2 is a subset of $S_1 + g_1$. Let D_1 be a maximum connected domain of S_2. Also $S - g_2$ is the sum of two mutually separated point sets S_3 and S_4. One of these, say S_3, is such that g_1 is a subset of $S_3 + g_2$. Let D_2 be a maximum connected domain of S_4. Let D_3 be a maximum connected domain of $g_1 + g_2$ distinct from D_1 or D_2. We know D_3 exists, since by hypotheses each of the continua g_1 and g_2 separates some complementary domain of the other. The domains D_1, D_2, and D_3 are then separated by $g_1 + g_2$ symmetrically with respect to D_3.

LEMMA 2. *If g and h are two continua which do not cross and which have no complementary domain in common, and if D is a complementary domain of g such that h contains a point of D and is a subset of $\overline{D}$, then $S-\overline{D}$ lies in a connected complementary domain of h.*

PROOF. Suppose D_1 and D_2 are two distinct complementary domains of h, each of which contains a point of $S-\overline{D}$. One of these domains does not contain a point of g, since g and h do not cross. Suppose D_2 contains no point of g. Then D contains no point of D_2 and therefore contains no point of the boundary of D_2. Since, by hypothesis, every point of h is in $D+g$, it follows that every point of the boundary of D_2 is in g as well as in h, and therefore is in $g \cdot h$. Since we have supposed that g contains no point of D_2, h and g have the complementary domain D_2 in common, and this contradicts a hypothesis of the lemma. We arrive at a similar contradiction if we suppose that g contains no point of D_1.

Notation. In what follows, if h_1 and h_2 are any two continua whose sum separates the points P_1, P_2, and P_3 symmetrically with respect to P_3, then D_{h1} and D_{h2} will designate the complementary domains of h_1+h_2 which contain P_1 and P_2, respectively. A similar notation will be used if h is replaced by any other symbol.

LEMMA 3. *If (1) g_1, g_2, h_1, and h_2 are four continua no one of which crosses another or has a complementary domain in common with another one of the four, (2) both g_1+g_2 and h_1+h_2 separate the points P_1, P_2, and P_3 symmetrically with respect to P_3, and (3) $h_1 \cdot h_2$ contains a point not in g_1+g_2, then the domain $S_1 = S-(h_1+h_2+D_{h1}+D_{h2})$ is a proper subset of the domain $S_2 = S-(g_1+g_2+D_{g1}+D_{g2})$ and h_1+h_2 contains $g_1 \cdot g_2$.*

PROOF. By hypothesis (3), $h_1 \cdot h_2$ contains a point P of some complementary domain of g_1+g_2. We know P cannot be in D_{g1}, for if so, then by Lemma 2, $S-\overline{D}_{g1}$, and therefore P_2+P_3, lies in a connected complementary domain E_2 of h_2. Since h_1+h_2 separates P_1, P_2, and P_3 symmetrically with respect to P_3, $\overline{E}_2$ does not contain h_1 (see Definition 2). Let E_1 designate the complementary domain of h_2 which with its boundary contains h_1. By Lemma 2, $S-\overline{E}_1$, which contains E_2, lies in a connected complementary domain E_3 of h_1. That is, the complementary domain E_2 of h_2 which contains P_2+P_3 is a subset of the complementary domain E_3 of h_1. Then h_1+h_2 does not separate P_2 from P_3, and we have a contradiction of a hypothesis of the lemma. It follows that $h_1 \cdot h_2$ cannot contain a point of D_{g1} and similarly cannot contain a point of D_{g2}.

Then $h_1 \cdot h_2$ must contain a point of S_2. Since D_{g1} and D_{g2} contain

P_1 and P_2, respectively, and since h_1 separates P_1 from P_2 and contains no point of $D_{g1}+D_{g2}$, it follows that D_{g1} and D_{g2} lie in mutually exclusive complementary domains of h_1. There are points Q_1 and Q_2 on the boundaries of D_{g1} and D_{g2} which are in g_1 and g_2, respectively, but which are not points of h_1, since h_1 and g_1, and similarly h_1 and g_2, have no complementary domain in common. Then there exist mutually exclusive complementary domains F_1 and F_2 of h_1 such that g_1 and g_2 are subsets of $\overline{F}_1$ and $\overline{F}_2$, respectively. Consequently $g_1 \cdot g_2$ is a subset of $\overline{F}_1 \cdot \overline{F}_2$, that is, of h_1. Therefore, as required, h_1+h_2 contains $g_1 \cdot g_2$. By hypothesis, D_{h1} and D_{h2} contain P_1 and P_2, respectively, and the sets $\overline{D}_{h1}$ and $\overline{D}_{h2}$ contain $D_{g1}+g_1$ and $D_{g2}+g_2$, respectively. Therefore S_1 is a subset of S_2; and S_1 is a proper subset of S_2, since by assumption S_2 contains a point of $h_1 \cdot h_2$.

THEOREM. *Let G be a collection of continua having the following properties*: (1) *No continuum of G crosses or contains a complementary domain in common with any other continuum of the collection.* (2) *Each continuum of G separates S and also separates some complemenetary domain of each other continuum in the collection. Then there exists a countable collection H of G such that each point common to two continua of the collection G is contained in some continuum of the collection H.**

PROOF. Since S is separable, there exists a countable subset K of S which has the property that every point of S is a point of K or a limit point of K. Let L be a countable collection of the permutations of the points of K taken three at a time. Let p be any permutation $P_1P_2P_3$ of the collection L. Let W_p be the collection of all sets each of which consists of two continua a and b of the collection G such that $a \cdot b$ is not vacuous and such that $a+b$ separates P_1, P_2, and P_3 symmetrically with respect to P_3. In accordance with the notation used above, if $w_\theta = a_\theta + b_\theta$ is any set of the collection W_p, then $D_{\theta 1}$ and $D_{\theta 2}$ will designate the complementary domains of $a_\theta + b_\theta$ which contain P_1 and P_2, respectively, and S_θ will designate the domain $S-(D_{\theta 1}+D_{\theta 2}+a_\theta+b_\theta)$. A similar notation will be used if w_θ is replaced by any similar symbol. Let α be a well ordered sequence of the sets of the collection W_p. Let β be a well ordered subsequence $v_1, v_2, \cdots, v_\omega, v_{\omega+1}, \cdots$ of α. If $v_\theta = w_\phi$, let c_θ and d_θ designate a_ϕ and b_ϕ, respectively. We shall define v_1 as being the first element of α. Let v_2 be the first element of α which follows v_1 and has the property

---

* That the set of points each of which is common to two members of the collection G is not necessarily countable may be seen if, in the cartesian plane, we define G as being the collection of all continua, each of which is the sum of the X axis between two points A and B and two vertical rays extending upward from A and B.

that $c_2 \cdot d_2$ contains a point not in $c_1 + d_1$. In general, if γ is any ordinal number such that v_ψ is defined for each ordinal number ψ which precedes γ, let v_γ be the first element of α which follows every v_ψ and has the property that $c_\gamma \cdot d_\gamma$ contains a point not in any $c_\psi + d_\psi$. By Lemma 3 it follows that if θ and ϕ are any two ordinal numbers such that θ precedes ϕ, then $c_\phi + d_\phi$ contains $c_\theta \cdot d_\theta$ and S_ϕ is a proper subset of S_θ. Furthermore if w_π is any element whatever of the sequence α, then by virtue of the definition of the subsequence β, it follows that $a_\pi \cdot b_\pi$ is a subset of some set of the sequence β. By a well known property of abstract sets, since S is separable, it follows that the sequence β is countable. Let H_p designate the countable collection of continua

$$c_1, d_1, c_2, d_2, \cdots, c_\omega, d_\omega, c_{\omega+1}, d_{\omega+1}, \cdots.$$

Let H be the collection of all continua of the collection G which are in H_p for some permutation p of the collection L. Then since H_p is countable for each p and since L is countable, it follows that H is countable.

Now let h_1 and h_2 be any two elements of the collection G which have a point in common. Since S is separable and locally connected, it follows from Lemma 1 that there exists a permutation p, $P_1 P_2 P_3$, of L such that $h_1 + h_2$ separates P_1, P_2, and P_3 symmetrically with respect to P_3. Consequently $h_1 + h_2$ belongs to W_p and $h_1 \cdot h_2$ is a subset of some continuum of G which belongs to H_p and therefore is a subset of some continuum of G which belongs to the countable collection H. The theorem is therefore true.

Corollaries. The following are obvious corollaries to the theorem:

Corollary 1. *If G is any collection of continua satisfying the hypotheses of the theorem, then there exists a subcollection G' of mutually exclusive continua of G which contains all but a countable number of the continua of G.*

Corollary 2. *If G is a collection of continua in the euclidean plane, each of which is an open curve or a simple closed curve, and if no one of these continua crosses any other one of the collection, then there exists a countable subcollection H of G such that every point common to two elements of the collection G is in some element of the collection H.*

Examples. We see that the condition that the elements of G do not cross is necessary if we let G consist of all straight lines in the euclidean plane. It is to be noted that under these circumstances no two elements of G have a complementary domain in common but

each two nonparallel elements of G cross each other. Obviously the conclusions of the theorem do not hold.

The following example will show that the condition that no two elements of the collection G shall have a complementary domain in common is also necessary. In the cartesian plane let M be a circle of radius 1 and center at the origin, and N a circle of radius 1 and center at the point (5, 5). Let G_1 be a collection which contains each continuum which is the sum of M and a horizontal straight line interval of length 10 whose left-hand end point is on the circle M and which contains no point within M. Let G_2 be a collection which contains each continuum which is the sum of N and a vertical straight line interval of length 10 whose upper end point is on the circle N and which contains no point within N. Let $G = G_1 + G_2$. No element of G crosses any other element of G, but uncountably many have a complementary domain in common with some other element of the collection. However, it is evident that no countable subcollection of G covers the set of points each of which is common to two continua of the collection G.

It is not known whether or not the condition that each element of G shall separate some complementary domain of every other one can be omitted.

OKLAHOMA AGRICULTURAL AND MECHANICAL COLLEGE

A PRINCIPAL AXIS TRANSFORMATION FOR NON-HERMITIAN MATRICES

CARL ECKART AND GALE YOUNG

The availability of the principal axis transformation for hermitian matrices often simplifies the proof of theorems concerning them. In working with non-hermitian matrices (square or rectangular) it was found that a generalization of this transformation has a similar use for them.* A special case of this generalization has been investigated by Sylvester† who proved Theorem 1 (below) for square matrices with real elements. The unitary matrices U and V are in that case orthogonal matrices with real elements. Special cases had also been

---

    * C. Eckart, *The kinetic energy of polyatomic molecules*, Physical Review, vol. 46 (1934), p. 383; C. Eckart and G. Young, *The approximation of one matrix by another of lower rank*, Psychometrika, vol. 1 (1936), p. 211; A. S. Householder and G. Young, *Matrix approximation and latent roots*, American Mathematical Monthly, vol. 45 (1938), p. 302.

    † Sylvester, Messenger of Mathematics, vol. 19 (1889), p. 42.