SHORTER NOTICES

Some Notes on Least Squares. By W. E. Deming. Washington, D.C., The Graduate School, Department of Agriculture, 1938. 181 pp.

The author is conservative in calling this work *Some Notes on Least Squares.* It may not be what one would call a finished product judged by the standards of textbook requirements, but a reading will show it to be much more than the skeleton outline of a series of lectures on least squares.

At the outset we find what he calls *fragments in the history of least squares* which presents a fair picture of its development from Gauss' time, and with the copious references throughout the text one gets a thumbnail bibliography of the subject brought up to the present.

The treatment of the subject is logical and consistent throughout. It may be likened to what students of harmony call "variations on a theme"; and the author's theme is *The Minimizing of the Sum of the Squared Residuals* as enunciated by Gauss in his "theoris motus" (1809). It is interesting to note that in all the problems in least squares considered he begins the attack with this principle, adapting it to the special case by simple mathematical stages. In fact, he almost apologizes for the necessary use of the first derivative!

Believing that, in the present day, least squares cannot be disassociated from statistical researches that have been made since Gauss, the author has woven some of it into the material comprising these notes.

It is regrettable that this method of treatment does not suggest to the reader that, after all, the whole scheme is based on probability; that no mention is made of the probable error (as a modification of mean square error) still used by a considerable number of research and technical men in connection with the normal curve.

After a few simple applications in finding the best value from several direct measures, the author takes up the General Problem in Least Squares for which he gives a complete solution, whether it is a problem in curve fitting with adjustable parameters or one involving geometrical conditions or a combination of both cases, with all measured quantities subject to error.

The solution is made possible, of course, by approximation, since approximate values of the unknowns are necessary and the application of the principle of Gauss leads up to the normal equations involving corrections to the approximations as unknowns. There is nothing new in this idea, but most problems result in equations which are either solved with great difficulty or else defy analysis so that a clarifying device must be used if definite results are hoped for.

Commenting on this phase the author states—"All problems in least squares theoretically can be solved without the use of Lagrange multipliers. Occasionally it may be easier to dispense with them, but the truth is that most problems become hopelessly involved. The elegance and simplicity that they lend to all problems seem to me sufficient to displace all other possible methods in the design of a routine procedure. If Kummell in 1879 had introduced Lagrange multipliers, he would surely have accomplished the general solution that he was looking for."

This solution may be easily adjusted to any set of conditions simple or complex: for instance, if some of the measured quantities are free from error, this is taken care of by the simple expedient of making their respective weights infinite.

In the section on curve fitting the plan is outlined for obtaining the best values for the parameters, their weights and errors, the adjustment of the observations, and the confidence belts associated with the curve.

Lastly there are interesting exercises and notes on the formation of the normal equations for various functions to best fit certain measurements such as straight line, parabola, exponential, exponential with a lineal component, the generalized hyperbola together with three examples in curve fitting completely worked out, viz.:

Example 1, fitting an isotherm $y=a+bx+cx^2+dx^4$ with parameters subject to the condition $x=1$ when $y=1$.

Example 2, the polynomial $y=a+bx+cx^2$ with both x and y subject to error.

Example 3, an example useful in forestry, fitting $x=ay^bz^c$ where x = volume of a tree, y = merchantable height of tree, and z = diameter at breast height, the data having been secured from 66 trees.

This treatise is to be commended for its completeness and for maintaining the single purpose of all work in least squares, the minimizing of $\sum\overline{\text{res}}^2$. Although the whole scheme is non-rigorous except for linear functions, the error introduced by ignoring the higher powers of the corrections to the assumed values for the parameters or other unknowns does not seriously affect the result. It seems to be the basis for a very good graduate course and shows clearly how the use of Lagrange multipliers may sometimes clarify an otherwise hopeless analytical problem.

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Les Definitions Modernes de la Dimension. By Georges Bouligand. Paris, Hermann, 1935. 44 pp.

This volume opens with a brief discussion of linear or affine geometry based on postulating the existence of an abstract system of elements called vectors which combine amongst themselves and with the real numbers to give us new elements of the same class. The dimension is determined by the number of linearly independent vectors. There is a brief discussion of spaces which in this sense would be of infinitely many dimensions.

Then denying the right of geometry to limit itself merely to spaces of this category, the author discusses three types of definition for dimension:

(1) The definitions which follow the direction of Fréchet who is interested in the spaces introduced by the needs of General Analysis. Fréchet develops a theory of dimension types or a topological frame based on the notion of homeomorphism between sets. His theory has as its axiomatic substructure three simple axioms concerning the closure of point sets.

(2) Those definitions where the fundamental notion is that of *separation* of sets by various subsets. These definitions emanate from the influence of Poincaré who started from the seeming contradiction of the existence of a (1-1) correspondence between the points of a line to which we should wish to assign the dimension one and the points of a plane which should be of dimension two. This line of development runs from Poincaré through Brouwer to Menger and Urysohn. The whole theory is based upon the three axioms fundamental in the Fréchet theory plus two additional axioms, that of normality and the second countability axiom, that is, a space in which distance can be defined. The author discusses the theorem of Lebesgue concerning intersection of coverings which can be used to give a definition of dimension independent of recurrence and which is usually used to show that the Menger-Urysohn definition gives the dimension n to sets to which we would intuitively assign the same dimension. A brief discussion is also given of the combinatorial point of view with respect to dimension which has been so ably developed by the powerful methods of Alexandroff.