RELATIONS AMONG THE FUNDAMENTAL SOLUTIONS OF THE GENERALIZED HYPERGEOMETRIC EQUATION WHEN $p=q+1$

II. LOGARITHMIC CASES*

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1. **Introduction.** In a previous paper [1], the author gave the relations among the non-logarithmic solutions of the equation

$$(1)\qquad \left\{\prod_{t=1}^{q+1}(\theta + a_t) - \frac{1}{z}\prod_{t=1}^{q+1}(\theta + c_t - 1)\right\} y = 0$$

where $\theta = z(d/dz)$ and where the a_t and c_t are any constants, real or complex, the only restriction being that one of the c_t must be equal to unity. If no two of the a_t or c_t are equal or differ by an integer, then equation (1) has $q+1$ linearly independent solutions about the point $z=0$ which may be written

$$(2)\qquad Y_{0j} = z^{1-c_j}\prod_{t=1}^{q+1}\frac{\Gamma(1+c_t-c_j)}{\Gamma(1+a_t-c_j)}\sum_{n=0}^{\infty}\prod_{t=1}^{q+1}\frac{\Gamma(1+a_t-c_j+n)}{\Gamma(1+c_t-c_j+n)} z^n,$$
$$j = 1, 2, \cdots, q+1; \ |z| < 1,$$

and $q+1$ linearly independent solutions about the point $z=\infty$ which may be written

$$(3)\qquad Y_{\infty j} = z^{-a_j}\prod_{t=1}^{q+1}\frac{\Gamma(1-a_t+a_j)}{\Gamma(1-c_t+a_j)}\sum_{n=0}^{\infty}\prod_{t=1}^{q+1}\frac{\Gamma(1-c_t+a_j+n)}{\Gamma(1-a_t+a_j+n)}\frac{1}{z^n},$$
$$j = 1, 2, \cdots, q+1; \ |z| > 1.$$

If, however, we assume that

$$c_2 - c_1 = l_1;\ c_3 - c_2 = l_2;\ \cdots;\ c_r - c_{r-1} = l_{r-1}$$

where each l_v is zero or a positive integer and assume at the same time that none of these r c_t is equal to or differs from any of the a_t by an integer, then the author has shown that the first r of the solutions (2) are replaced by the following forms [2]:

$$(4)\qquad Y_{0j} = \sum_{v=1}^{j} z^{1-c_v}\frac{(j-1)!}{(j-v)!}\left[\frac{\partial^{j-v}}{\partial w^{j-v}} z^w G_v(w, z)\right]_{w=0},$$
$$j = 1, 2, \cdots, r; \ |z| < 1,$$

where

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$$G_v(w, z) = (-1)^{1-v-\sum_{t=1}^{v-1} t l_t}\left(\frac{\pi w}{\sin \pi w}\right)^{1-v} \prod_{t=1}^{q+1} \frac{\Gamma(1+c_t-c_1+w)}{\Gamma(1+a_t-c_1+w)}$$

$$(5) \qquad \cdot \sum_{n=0}^{l_{v-1}-1} \prod_{t=1}^{v-1} \Gamma(c_v - c_t - w - n)\Gamma(1 + a_t - c_v + w + n)$$

$$\cdot \prod_{t=v}^{q+1} \frac{\Gamma(1+a_t-c_v+w+n)}{\Gamma(1+c_t-c_v+w+n)} [(-1)^{v-1}z]^n$$

in which the -1 factor and the first product of the summation are missing when $v=1$, and in which we make the special definition $l_0=\infty$; moreover, we make the special convention that $G_v(w, z)=0$ if $l_{v-1}=0$. Similarly, if we assume that $a_1-a_2=k_1$; $a_2-a_3=k_2$; $\cdots$; $a_{s-1}-a_s=k_{s-1}$ where each k_v is zero or a positive integer and assume at the same time that none of these s a_t is equal to or differs from any of the c_t by an integer, then the first s of the solutions (3) are replaced by the following forms [2]:

$$(6) \qquad Y_{\infty j} = \sum_{v=1}^{j} z^{-a_v} \frac{(j-1)!}{(j-v)!}\left[\frac{\partial^{j-v}}{\partial w^{j-v}} z^{-w} F_v(w, z)\right]_{w=0},$$

$$j = 1, 2, \cdots, s; \ |z| > 1,$$

where

$$F_v(w, z) = (-1)^{1-v-\sum_{t=1}^{v-1} t k_t}\left(\frac{\pi w}{\sin \pi w}\right)^{1-v} \prod_{t=1}^{q+1} \frac{\Gamma(1-a_t+a_1+w)}{\Gamma(1-c_t+a_1+w)}$$

$$(7) \qquad \cdot \sum_{n=0}^{k_{v-1}-1} \prod_{t=1}^{v-1} \Gamma(a_t - a_v - w - n)\Gamma(1 - c_t + a_v + w + n)$$

$$\cdot \prod_{t=v}^{q+1} \frac{\Gamma(1-c_t+a_v+w+n)}{\Gamma(1-a_t+a_v+w+n)}\left[(-1)^{v-1}\frac{1}{z}\right]^n$$

in which we make special conventions of the same type as those made in connection with (5).

It is the purpose of this paper to develop the relations among the solutions of (1) when one or both of the two sets of solutions contain logarithmic members. The results of this paper generalize those of Mehlenbacher [3] and Lindelöf [4] who treated the case in which $q=1$.

2. **All Y_{0j} non-logarithmic.** In this case we may state the following theorem:

THEOREM 1. *If all of the solutions Y_{0j} of equation* (1) *are non-logarithmic in character while the first s of the solutions $Y_{\infty j}$ are logarith-*

mic, then the Y_{0j}, when extended analytically outside their circle of convergence, may be expressed linearly in terms of the $Y_{\infty j}$ in the following forms, it being understood throughout that $0<\arg z<2\pi$:

$$
\begin{aligned}
Y_{0j} = {} & \frac{e(a_s - c_j, 1)}{(s-1)!} \prod_{t=1}^{q+1} \frac{\Gamma(1 + c_t + c_j)}{\Gamma(1 + a_t - c_j)} \sum_{k=1}^{s} C_{s-1,k-1} P_j^{(s-k)}(0, 1) Y_{\infty k} \\
& + \sum_{k=s+1}^{q+1} \left\{ e(a_k - c_j + 1, 1) \frac{\Gamma(c_j - a_k)\Gamma(1 + c_k - c_j)}{\Gamma(c_k - a_k)} \right. \\
& \left. \cdot \prod_{t=1, t\neq k}^{q+1} \frac{\Gamma(a_t - a_k)\Gamma(1 + c_t - c_j)}{\Gamma(c_t - a_k)\Gamma(1 + a_t - c_j)} Y_{\infty k} \right\}, \\
& \qquad j = 1, 2, \cdots, q+1,
\end{aligned} \tag{8}
$$

where $e(a_s - c_j, 1)$ and $e(a_k - c_j + 1, 1)$ are to be obtained from the definition

$$e(w, m) = \exp \left\{ (1/2) i\pi [1 - (-1)^m] w \right\}$$

and where $P_j^{(s-k)}(0, 1)$ denotes the $(s-k)$th derivative with respect to w of the function

$$
\begin{aligned}
P_j(w, m) = {} & (-1)^{s + \sum_{t=1}^{s-1}(s+t+m)k_t} \cdot e(w, m) \left(\frac{\pi w}{\sin \pi w} \right)^s \\
& \cdot \prod_{t=1}^{s} \frac{\Gamma^m(c_j - a_1 - w)\Gamma^m(w - c_j + a_1 + 1)}{\Gamma(c_t - a_1 - w)\Gamma(w - a_t + a_1 + 1)} \\
& \cdot \prod_{t=1}^{q+1} \frac{\Gamma(a_t - a_1 - w)}{\Gamma(c_t - a_1 - w)}
\end{aligned} \tag{9}
$$

*evaluated for $w = 0$ and $m = 1$.**

PROOF. The proof follows the same outline as that employed in the non-logarithmic cases [1]. Under present assumptions, the function

$$g(w) = \prod_{t=1}^{q+1} \frac{\Gamma(1 + a_t - c_j + w)}{\Gamma(1 + c_t - c_j + w)} \tag{10}$$

continues to have simple poles at the points

$$w = c_j - a_k - n - 1, \quad n = 0, 1, 2, \cdots; \; k = s+1, \cdots, q+1, \tag{11}$$

but now has poles of order $s - v + 1$ at the points

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* Throughout this paper, an expression of the form $P^{(k)}(w)$ indicates the kth derivative of $P(w)$ while an expression of the form $P^k(w)$ indicates the kth power of $P(w)$.

(12) $w = c_j - a_v - n - 1,\ n = 0, 1, 2, \cdots, k_{v-1} - 1; v = 1, 2, \cdots, s.$

Therefore, in the result previously obtained,* we must replace the terms for $k=1, 2, \cdots, s$ by

$$- z^{1-c_j}\prod_{t=1}^{q+1}\frac{\Gamma(1 + c_t - c_j)}{\Gamma(1 + a_t - c_j)}\sum_{v=1}^{s}\sum_{n=0}^{k_{v-1}-1} R_{n,v} \tag{13}$$

where $R_{n,v}$ denotes the residue of the function

$$\frac{\pi(- z)^w g(w)}{\sin \pi w} = (- z)^w\Gamma(w)\Gamma(1 - w)\prod_{t=1}^{q+1}\frac{\Gamma(1 + a_t - c_j + w)}{\Gamma(1 + c_t - c_j + w)} \tag{14}$$

at the point $w=c_j-a_v-n-1$.

In order to compute $R_{n,v}$, consider

$$R_{n,v} = \frac{1}{2\pi i}\int_{C_n}(- z)^w\Gamma(w)\Gamma(1 - w)\prod_{t=1}^{q+1}\frac{\Gamma(1 + a_t - c_j + w)}{\Gamma(1 + c_t - c_j + w)}dw \tag{15}$$

where C_n^v surrounds the pole $w=c_j-a_v-n-1$ of (14) but no other pole of (14). If in (15), we replace w by $-(w-c_j+a_v+n+1)$, we get

$$\begin{aligned} R_{n,v} = &- \frac{(- z)^{c_j-a_v-n-1}}{2\pi i}\int_{C_0}(- z)^{-w}\Gamma(c_j - a_v - w - n - 1) \\ &\cdot\Gamma(w - c_j + a_v + n + 1)\prod_{t=1}^{q+1}\frac{\Gamma(a_t - a_v - w - n)}{\Gamma(c_t - a_v - w - n)}dw \end{aligned} \tag{16}$$

where C_0 surrounds the origin. By several applications of the relation

$$\Gamma(1 - w) = \frac{\pi}{\Gamma(w) \sin \pi w} \tag{17}$$

we may change (16) to the form

$$\begin{aligned} R_{n,v} = &- \frac{z^{c_j-a_v-1}e(a_s - c_j, 1)(- 1)^{1-v-\sum_{t=1}^{v-1}tk_t}}{[(- 1)^{v-1}z]^n 2\pi i} \\ &\cdot\int_{C_0}\frac{z^{-w}P_j(w, 1)}{w^{s-v+1}}\left(\frac{\pi w}{\sin \pi w}\right)^{1-v}\prod_{t=1}^{q+1}\frac{\Gamma(1 - a_t + a_1 + w)}{\Gamma(1 - c_t + a_1 + w)} \\ &\cdot\prod_{t=1}^{v-1}\Gamma(a_t - a_v - w - n)\Gamma(1 - c_t + a_v + w + n) \\ &\cdot\prod_{t=v}^{q+1}\frac{\Gamma(1 - c_t + a_v + w + n)}{\Gamma(1 - a_t + a_v + w + n)}dw. \end{aligned} \tag{18}$$

* See [1], equation (5).

Therefore,

$$(19)\quad \sum_{n=0}^{k_{v-1}-1} R_{n,v} = - z^{c_j - a_{v-1}}e(a_s - c_j, 1) \int_{C_0} \frac{z^{-w}P_j(w, 1)F_v(w, z)}{w^{s-v+1}}\,dw.$$

When (19) is evaluated by the theorems of the calculus of residues [5], we get

$$(20)\quad \sum_{n=0}^{k_{v-1}-1} R_{n,v} = - \frac{z^{c_j - a_v - 1}e(a_s - c_j, 1)}{(s-v)!} \left[\frac{\partial^{s-v}}{\partial w^{s-v}} z^{-w}P_j(w, 1)F_v(w, z) \right]_{w=0}.$$

Therefore

$$(21)\quad \begin{aligned} \sum_{v=1}^{s} \sum_{n=0}^{k_{v-1}-1} R_{n,v} &= - z^{c_j-1}e(a_s - c_j, 1) \sum_{v=1}^{s} \frac{z^{-a_v}}{(s-v)!} \\ &\quad \cdot \sum_{h=0}^{s-v} C_{s-v,h}P_j^{(h)}(0, 1) \left[\frac{\partial^{s-v-h}}{\partial w^{s-v-h}} z^{-w}F_v(w, z) \right]_{w=0} \\ &= - \frac{z^{c_j-1}e(a_s - c_j, 1)}{(s-1)!} \sum_{k=1}^{s} C_{s-1,k-1}P_j^{(s-k)}(0, 1)Y_{\infty k}. \end{aligned}$$

Substituting (21) into (13), we get the final form of the replacements to be made in the result previously obtained. When these replacements are made, we obtain the desired result (8).

In a previous paper [1], the author gave the expression for the non-logarithmic $Y_{\infty j}$ in terms of the Y_{0j}. It remains, then, to develop the expression for the logarithmic $Y_{\infty j}$ in terms of the Y_{0j}. In this connection, we may state the following theorem:

THEOREM 2. *Under the conditions stated in Theorem* 1, *the logarithmic* $Y_{\infty j}$, *when extended analytically outside their circle of convergence, may be expressed linearly in terms of the* Y_{0j} *in the following forms, it being understood throughout that* $0 < \arg(1/z) < 2\pi$:

$$(22)\quad \begin{aligned} Y_{\infty j} = - \sum_{k=1}^{q+1} \sum_{m=1}^{j} \Bigg\{ &(m-1)!K_{mj}(L)e(c_k - a_m, m) \\ &\cdot \frac{\Gamma^m(c_k - a_m)\Gamma^m(1 - c_k + a_m)}{\Gamma(c_k - a_k)} \prod_{t=1, t\neq k}^{q+1} \frac{\Gamma(c_k - c_t)}{\Gamma(c_k - a_t)} Y_{0k} \Bigg\}, \end{aligned}$$

$$j = 1, 2, \cdots, s,$$

where $K_{mj}(L)$ *is the quotient of the cofactor of the element in the mth row and jth column of the determinant*

$$
(23)\quad \Delta_j(L) = \begin{vmatrix} L_1(0) & 0 & 0 & \cdots & 0 \\ L_2'(0) & L_2(0) & 0 & \cdots & 0 \\ L_3''(0) & 2L_3'(0) & L_3(0) & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ L_j^{(j-1)}(0) & C_{j-1,1}L_j^{(j-2)}(0) & C_{j-1,2}L_j^{(j-3)}(0) & \cdots & L_j(0) \end{vmatrix}
$$

and $\Delta_j(L)$ itself, the $L_j(w)$ being defined by the formula

$$
(24)\qquad L_j(w) = e(w, j)\left(\frac{\pi w}{\sin \pi w}\right)^j \prod_{t=1}^{q+1} \frac{\Gamma(1 - c_t + a_1 + w)}{\Gamma(1 - a_t + a_1 + w)}
$$

and the cofactor of $L_1(0)$ in $\Delta_1(L)$ being defined equal to unity.

PROOF. In order to prove this theorem, we follow the same procedure as that used in the proof of a theorem due to Ford [6]. If in Ford's proof, we replace the integrand which he uses by

$$
(25)\qquad \frac{(\pm z)^{-w}\pi^j}{\sin^j \pi w} \prod_{t=1}^{q+1} \frac{\Gamma(1 - c_t + a_j + w)}{\Gamma(1 - a_t + a_j + w)}, \quad j = 1, 2, \cdots, s,
$$

where the upper or lower sign is to be taken according as j is even or odd, we obtain in place of Ford's final result

$$
(26)\qquad \begin{aligned} &\sum_{n=-\infty}^{\infty} \frac{1}{2\pi i}\int_{C_n} \frac{(\pm z)^{-w}\pi^j}{\sin^j \pi w} \prod_{t=1}^{q+1} \frac{\Gamma(1 - c_t + a_j + w)}{\Gamma(1 - a_t + a_j + w)} dw \\ &\qquad \sim - \sum_{k=1}^{q+1}\sum_{n=0}^{\infty} S_{n,k,j}, \quad j = 1, \cdots, s; 0 < \arg(1/z) < 2\pi, \end{aligned}
$$

where C_n surrounds the pole $w=n$ of (25) but no other pole of (25), and where $S_{n,k,j}$ denotes the residue of (25) at the point $w=c_k-a_j-n-1$ which, under the present assumptions, is a simple pole of (25). If we replace w by $w+n$ on the left in (26), we obtain

$$
(27)\qquad \begin{aligned} &\sum_{n=-\infty}^{\infty} \frac{z^{-n}}{2\pi i}\int_{C_0} \frac{(\pm z)^{-w}\pi^j}{\sin^j \pi w} \prod_{t=1}^{q+1} \frac{\Gamma(1 - c_t + a_j + w + n)}{\Gamma(1 - a_t + a_j + w + n)} dw \\ &= \sum_{n=1}^{\infty} \frac{z^{n}}{2\pi i}\int_{C_0} \frac{(\pm z)^{-w}\pi^j}{\sin^j \pi w} \prod_{t=1}^{q+1} \frac{\Gamma(1 - c_t + a_j + w - n)}{\Gamma(1 - a_t + a_j + w - n)} dw \\ &\quad + \sum_{v=1}^{j} z^{a_j-a_v} \sum_{n=0}^{k_{v-1}-1} \frac{z^{-n}}{2\pi i}\int_{C_0} \frac{(\pm z)^{-w}\pi^j}{\sin^j \pi w} \\ &\quad \cdot \prod_{t=1}^{q+1} \frac{\Gamma(1 - c_t + a_v + w + n)}{\Gamma(1 - a_t + a_v + w + n)} dw \end{aligned}
$$

where C_0 surrounds the origin. If we now apply (17) a number of times, the right side of (27) becomes

$$
\begin{aligned}
&\sum_{n=1}^{\infty} \frac{z^n(-1)^{jn+\sum_{t=1}^{j}(a_t-a_j-1)}}{2\pi i} \int_{C_0} (\pm z)^{-w} \\
&\quad \cdot \prod_{t=1}^{j} \Gamma(a_t - a_j - w + n)\Gamma(1 - c_t + a_j + w - n) \\
(28)\qquad &\quad \cdot \prod_{t=j+1}^{q+1} \frac{\Gamma(1 - c_t + a_j + w - n)}{\Gamma(1 - a_t + a_j + w - n)} dw \\
&\quad + \sum_{v=1}^{j} \frac{z^{a_j - a_v}}{2\pi i} \int_{C_0} \frac{z^{-w}L_j(w)F_v(w, z)}{w^{j-v+1}} dw.
\end{aligned}
$$

But since the integrand of the first integral of (28) is analytic for $w=0$, the first summation vanishes. When the second term of (28) is evaluated by the theorems of the calculus of residues [5], we obtain

$$
\begin{aligned}
(29)\qquad & z^{a_j}\sum_{v=1}^{j} \frac{z^{-a_v}}{(j-v)!}\left[\frac{\partial^{j-v}}{\partial w^{j-v}} z^{-w}L_j(w)F_v(w, z)\right]_{w=0} \\
&\qquad = \frac{z^{a_j}}{(j-1)!}\sum_{m=1}^{j} C_{j-1,m-1}L_j^{(j-m)}(0)Y_{\infty m}.
\end{aligned}
$$

As to the right member of (26), since $w=c_k-a_j-n-1$ is a simple pole of (25), we may compute the residue $S_{n,k,j}$ without difficulty. When this residue and (29) are used in (26), we obtain

$$
\begin{aligned}
&\sum_{m=1}^{j} C_{j-1,m-1}L_j^{(j-m)}(0)Y_{\infty m} \\
(30)\qquad &\sim -(j-1)!\sum_{k=1}^{q+1}\left\{e(c_k - a_j, j)\frac{\Gamma^j(c_k - a_j)\Gamma^j(1 - c_k + a_j)}{\Gamma(c_k - a_k)}\right. \\
&\qquad \left.\cdot \prod_{t=1, t\neq k}^{q+1} \frac{\Gamma(c_k - c_t)}{\Gamma(c_k - a_t)} Y_{0k}\right\}, \qquad j = 1, 2, \cdots, s.
\end{aligned}
$$

Since all $Y_{\infty m}$ and all Y_{0k} which appear in (30) are defined by series which converge for $|z|>1$, and $|z|<1$, respectively, we may replace $\sim$ by $=$ in (30). Although this gives us s equations, only the first j of these need be used in solving for a particular $Y_{\infty j}$. Solving these by Cramer's rule, we obtain the desired result (22).

3. **All $Y_{\infty j}$ non-logarithmic.** By means of proofs similar to those given in §2, we may establish the following theorems:

THEOREM 3. *If all of the solutions $Y_{\infty j}$ of equation* (1) *are non-logarithmic in character while the first r of the solutions Y_{0j} are logarithmic, then the logarithmic Y_{0j}, when extended analytically outside their circle of convergence, may be expressed linearly in terms of the $Y_{\infty j}$ in the following forms, it being understood throughout that* $0<\arg z<2\pi$:

$$
\begin{aligned}
Y_{0j} = -\sum_{k=1}^{q+1}\sum_{m=1}^{j}\Bigg\{&(m-1)!K_{mj}(N)e(a_k-c_m, m) \\
&\cdot\frac{\Gamma^m(c_m-a_k)\Gamma^m(1-c_m+a_k)}{\Gamma(c_k-a_k)}\prod_{t=1,t\neq k}^{q+1}\frac{\Gamma(a_t-a_k)}{\Gamma(c_t-a_k)}Y_{\infty k}\Bigg\}, \qquad (31)\\
&\qquad\qquad j=1, 2, \cdots, r,
\end{aligned}
$$

where $K_{mj}(N)$ is defined as in Theorem 2 with $L_j(w)$ replaced by

$$N_j(w) = e(-w, j)\left(\frac{\pi w}{\sin \pi w}\right)^j\prod_{t=1}^{q+1}\frac{\Gamma(1+a_t-c_1+w)}{\Gamma(1+c_t-c_1+w)}. \qquad (32)$$

THEOREM 4. *Under the conditions stated in Theorem* 3, *the $Y_{\infty j}$, when extended analytically outside their circle of convergence, may be expressed linearly in terms of the Y_{0j} in the following forms, it being understood throughout that* $0<\arg(1/z)<2\pi$:

$$
\begin{aligned}
Y_{\infty j} = &\frac{e(c_r-a_j, 1)}{(r-1)!}\prod_{t=1}^{q+1}\frac{\Gamma(1-a_t+a_j)}{\Gamma(1-c_t+a_j)}\sum_{k=1}^{r}C_{r-1,k-1}Q_j^{(r-k)}(0, 1)Y_{0k} \\
&+\sum_{k=r+1}^{q+1}\Bigg\{e(c_k-a_j-1, 1)\frac{\Gamma(c_k-a_j)\Gamma(1-a_k+a_j)}{\Gamma(c_k-a_k)} \qquad (33)\\
&\cdot\prod_{t=1,t\neq k}^{q+1}\frac{\Gamma(c_k-c_t)\Gamma(1-a_t+a_j)}{\Gamma(c_k-a_t)\Gamma(1-c_t+a_j)}Y_{0k}\Bigg\}, \\
&\qquad\qquad j=1, 2, \cdots, q+1,
\end{aligned}
$$

where $Q_j^{(r-k)}(0, 1)$ denotes the $(r-k)$th derivative with respect to w of the following function, evaluated for $w=0$ and $m=1$:

$$
\begin{aligned}
Q_j(w, m) = &(-1)^{r+\sum_{t=1}^{r-1}(r+t+m)l_t}e(-w, m)\left(\frac{\pi w}{\sin \pi w}\right)^r \qquad (34)\\
&\cdot\prod_{t=1}^{r}\frac{\Gamma^m(c_1-a_j-w)\Gamma^m(w-c_1+a_j+1)}{\Gamma(c_1-a_t-w)\Gamma(w-c_1+c_t+1)}\prod_{t=r+1}^{q+1}\frac{\Gamma(c_1-c_t-w)}{\Gamma(c_1-a_t-w)}.
\end{aligned}
$$

4. **Logarithmic members in both Y_{0j} and $Y_{\infty j}$.** The procedure used in the proof of Theorem 2 may be used again in this case. The only additional difficulties arise from the fact that the residue terms like $S_{n,k,j}$ of (26) do not all come from simple poles. But the steps in the

proofs of the following theorems are the same as those in the proof of Theorem 2.

THEOREM 5. *If the first r of the solutions Y_{0j} and the first s of the solutions $Y_{\infty j}$ of equation* (1) *are logarithmic in character, then the logarithmic Y_{0j}, when extended analytically outside their circle of convergence, may be expressed linearly in terms of the $Y_{\infty j}$ in the following forms, it being understood throughout that* $0<\arg z<2\pi$:

$$
\begin{aligned}
Y_{0j} = \sum_{m=1}^{j} (m-1)!K_{mj}(N)\Bigg\{ & \frac{e(a_s - c_m, m)}{(s-1)!} \sum_{k=1}^{s} C_{s-1,k-1} P_m^{(s-k)}(0,m) Y_{\infty k} \\
& - \sum_{k=s+1}^{q+1} e(a_k - c_m, m) \frac{\Gamma^m(c_m - a_k)\Gamma^m(1 - c_m + a_k)}{\Gamma(c_k - a_k)} \\
& \cdot \prod_{t=1, t\neq k}^{q+1} \frac{\Gamma(a_t - a_k)}{\Gamma(c_t - a_k)} Y_{\infty k} \Bigg\}, \qquad j = 1, 2, \cdots, r.
\end{aligned} \tag{35}
$$

THEOREM 6. *Under the conditions stated in Theorem* 5, *the logarithmic $Y_{\infty j}$, when extended analytically outside their circle of convergence, may be expressed linearly in terms of the Y_{0j} in the following forms, it being understood throughout that* $0<\arg(1/z)<2\pi$:

$$
\begin{aligned}
Y_{\infty j} = \sum_{m=1}^{j} (m-1)!K_{mj}(L)\Bigg\{ & \frac{e(c_r - a_m, m)}{(r-1)!} \sum_{k=1}^{r} C_{r-1,k-1} Q_m^{(r-k)}(0,m) Y_{0k} \\
& - \sum_{k=r+1}^{q+1} e(c_k - a_m, m) \frac{\Gamma^m(c_k - a_m)\Gamma^m(1 - c_k + a_m)}{\Gamma(c_k - a_k)} \\
& \cdot \prod_{t=1, t\neq k}^{q+1} \frac{\Gamma(c_k - c_t)}{\Gamma(c_k - a_t)} Y_{0k} \Bigg\}, \qquad j = 1, 2, \cdots, s.
\end{aligned} \tag{36}
$$

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