# ON A CERTAIN CLASS OF SYMMETRIC HYPERSURFACES

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In the literature of algebraic geometry, relatively little is written on hypersurfaces $H$ of order $n$ and of $r-1$ dimensions in $S_r$, invariant under the symmetric permutation group $G$ on $r+2$ homogeneous coordinates whose sum is zero. Surfaces in $S_3$ invariant under the symmetric $G_{120}$ have been studied by Emch [1]; the Clebsch diagonal surface has been discussed by Clebsch [2], Eckardt [3], and Ciani [4, 5, 6]. The Segre cubic variety in $S_4$ has been investigated by Segre [8]; Ciani [7] has developed properties of loci in $S_4$ invariant under the $G_{720}$.

It is well known that the equation of such a hypersurface $H$ is ex-expressible uniquely in terms of the elementary symmetric functions $p_i$ (of order $i$ in $x_1, \cdots, x_{r+2}$), or in terms of the $\Sigma$ functions $\Sigma_n = \sum_{i=1}^{r+2} x_i^n$. With the condition $\Sigma_1 = 0$, any $H$ is a member of the linear system

$$\sum_{i=1}^{N} C_i \Sigma_2^a \Sigma_3^b \cdots \Sigma_{r+2}^d = 0 \tag{1}$$

where $N$ is the number of nonnegative solutions of the Diophantine equation

$$2a + 3b + \cdots + (r+2)d = 0. \tag{2}$$

It follows immediately that there is a unique hyperquadric $H_2$, $\Sigma_2 = 0$ or $p_2 = 0$, a unique cubic $H_3$, $\Sigma_3 = 0$ or $p_3 = 0$, a pencil of quartics $H_4$, $\lambda\Sigma_4 + \mu\Sigma_2^2 = 0$, a pencil of quintics $H_5$, $\lambda\Sigma_5 + \mu\Sigma_2\Sigma_3 = 0$, and so on, in $S_r$ invariant under $G$.

Since there are no real points on $\Sigma_n = 0$ if $n$ is even, values of $n$ will be restricted to odd positive integers throughout the remainder of this paper.

Emch [1] has shown that the equation of any surface of odd order in $S_3$, invariant under the symmetric $G_{120}$ on 5 $x$'s whose sum is zero necessarily has the form $Ap_3 + Bp_5 = 0$, which is equivalent to $\lambda\Sigma_3 + \mu\Sigma_5 = 0$. The obvious generalization of this statement is that any $H$ of order $n$ necessarily is a member of the linear system

$$A\Sigma_3 + B\Sigma_5 + \cdots + D\Sigma_{r+2} = 0; \tag{3}$$

if $r = 2k+1$, and in case $r = 2k$, of the system

$$A\Sigma_3 + B\Sigma_5 + \cdots + C\Sigma_{r+1} = 0. \tag{4}$$

If $r=2k+1$, all $H$ of (3) contain the base $\Sigma_3=\Sigma_5=\cdots=\Sigma_{r+2}=0$. This base is of order $\alpha=(r+2)\cdot r!/2^k\cdot k!$ and of dimension $k$, consisting of the $\alpha$ subspaces into which the $S_k$ $[x_1+x_2=0, x_3+x_4=0, \cdots, x_r+x_{r+1}=0, x_{r+2}=0]$ is carried by the substitutions of $G$. Each $S_k$ is invariant under a subgroup of $G$ of order $2^{k+1}\cdot(k+1)!$.

If $r=2k$, all $H$ of (4) contain the base $\Sigma_3=\Sigma_5=\cdots=\Sigma_{r+1}=0$, of order $\beta=(r+1)!/2^k\cdot k!$ and of dimension $k$, consisting of the $\beta$ subspaces into which the substitutions of $G$ carry the $S_k$ $[x_1+x_2=0, x_3+x_4=0, \cdots, x_{r+1}+x_{r+2}=0]$; each $S_k$ is invariant under a subgroup of $G$ of order $2^{k+1}\cdot(k+1)!$.

The hypersurfaces $\Sigma_n=0$ have as double points only those points whose coordinates are proportional to $(n-1)$th roots of unity, since at a double point the $r+1$ partial derivatives $\partial\Sigma_n/\partial x_i=nx_i^{n-1}-nx_{r+2}^{n-1}$, $i=1, 2, \cdots, r+1$, must vanish (with the dependence of $x_{r+2}$ expressed by $\Sigma_1=0$). It follows immediately that $\Sigma_n=0$ has $C_{r+1,k}$ real double points if $r=2k$, and has no real double points if $r=2k+1$. Imaginary double points of $\Sigma_n=0$ will occur whenever there is a set of $r+2$ $(n-1)$th roots of unity, not all real, whose sum is zero.

If $r=2k+1$, then none of the $\alpha$ subspaces $S_k$ passes through a double point of $\Sigma_n=0$, since at a double point no $x_i=0$.

If $r=2k$, then in each of the $\beta$ subspaces $S_k$ there are $(n-1)^k$ double points of $\Sigma_n=0$, of which exactly $2^k$ are real; through each of the real double points pass $(k+1)!$ subspaces $S_k$. If $r=2k$, and $2k+1$ is a prime, then the $\beta$ subspaces $S_k$ divide into $(r-1)!$ sets of $(r+1)$-tuples, each $(r+1)$-tuple being transformable into another $(r+1)$-tuple by a cyclic substitution of $G$ of period $r+1$. (This is a generalization of the 6 quintuples of planes on the Segre cubic variety.)

The hypersurfaces $\Sigma_n=0$ can contain no points of multiplicity greater than two, since not all the partial derivatives of higher order vanish at any point.

Eckardt [3] has given an admirable synthetic and analytic discussion of the properties of an Eckardt point of a surface in $S_3$. The analytic generalization is immediate. Let a generalized Eckardt point $E$ of a hypersurface $F$, of order $m>2$ and of $r-1$ dimensions in $S_r$, be a simple point of $F$ such that the hyperplane $T$ tangent to $F$ in $E$ intersects $F$ in a hypercone with a vertex at $E$. We may so choose the polylateral of reference in $S_r$ that $E$ is $(1, 0, \cdots, 0)$, and $T$ is $x_2=0$. Then the equation of $F$ necessarily has the form

$$(5)\qquad x_2x_1^{m-1}+a_1x_2x_1^{m-2}+\cdots+a_{m-2}x_2x_1+a_m=0$$

where $a_i$ is a form of order $i$ in $x_2, x_3, \cdots, x_{r+1}$.

It follows immediately that the $i$th polar of $E$ reduces into $T$

($x_2=0$) and a hypersurface of order $m-i-1$ which does not pass through $E$.

Conversely, if the polar of a point $P$ with respect to a hypersurface $F$ of order $m$ reduces into a hyperplane $\pi$ passing through $P$ and a hypersurface of order $m-2$ not passing through $P$, then if $F$ does not reduce into $\pi$ and a hypersurface of order $m-1$, $P$ is a generalized Eckardt point of $F$ and $\pi$ is tangent to $F$ at the point $P$.

On each hypersurface $\Sigma_n=0$ in $S_r$ there are $C_{r+2,2}$ Eckardt points $E_{ij}$ ($x_i=-x_j\neq 0$; $x_s=0$, $s\neq i, j$), with the hyperplane $x_i+x_j=0$ tangent to $\Sigma_n=0$ at $E_{ij}$. The polar of $E_{ij}$ with respect to $\Sigma_n=0$ is $x_i^{n-1}-x_j^{n-1}=0$, which contains the hyperplanes $x_i+x_j=0$ and $x_i-x_j=0$. The latter is the axis of the perspectivity $(ij)$, with center at $E_{ij}$, under which $\Sigma_n=0$ and the polar of $E_{ij}$ are invariant. The $(n-1)$-fold locus $x_i=x_j=0$ of the polar of $E_{ij}$ contains $C_{r,2}$ Eckardt points $E_{st}$, $(s, t\neq i, j)$.

If $r=2k$, the hyperplane $x_i+x_j=0$ contains $C_{r,k}$ real double points of $\Sigma_n=0$, and $x_i-x_j=0$ contains the remaining $C_{r,k-1}$ real double points. Each real double point $D$ of $\Sigma_n=0$ is collinear with $(k+1)^2$ couples of points of $\Sigma_n=0$, each couple consisting of an $E_{ij}$ and the double point corresponding to $D$ under $(ij)$. In each of the $\beta$ subspaces $S_k$ on $\Sigma_n=0$ there are $k+1$ points $E_{ij}$, and through each $E_{ij}$ there pass $r!/2^k\cdot k!$ subspaces $S_k$, while the three collinear points $E_{ij}$, $E_{ik}$, $E_{jk}$ do not lie in a common $S_k$ on $\Sigma_n=0$. An $S_k$ is the locus of points common to the $k+1$ hyperplanes tangent to $\Sigma_n$ at the $k+1$ points $E_{ij}$ in the $S_k$.

If $r=2k+1$, through each $E_{ij}$ on $\Sigma_n=0$ pass $(r+1)!/2^k\cdot k!$ subspaces $S_k$, with $k+1$ points $E_{ij}$ in each $S_k$.

The Eckardt point of a hypersurface $F$ is in general of multiplicity $r-1$ on the Hessian of $F$; the Eckardt points of $\Sigma_n=0$ are of multiplicity $(r-1)(n-2)$ on the Hessian of $\Sigma_n=0$, whose equation is $\sum_{i=1}^{r+2} 1/x_i^{n-2}=0$.

## References

1. Arnold Emch, *Some geometric applications of the symmetric collineation groups on five variables*, Tôhoku Mathematical Journal, vol. 40 (1934), pp. 216–221.

2. A. Clebsch, *Ueber die Anwendung der quadratischen Substitution auf die Gleichungen 5ten Grades und die geometrische Theorie des ebenen Fünfseits*, Mathematische Annalen, vol. 4 (1871). See in particular pp. 331–340.

3. F. E. Eckardt, *Ueber diejenigen Flächen dritten Grades, auf denen sich drei gerade Linien in einem Punkte schneiden*, Mathematische Annalen, vol. 10 (1876), pp. 227–272.

4. Edgardo Ciani, *Sulla superficie diagonale di Clebsch*, Rendiconti della Accademia dei Lincei, vol. 7, no. 6.

5. ———, *Sul pentaedro complete*, Rendiconti della Accademia dei Lincei, vol. 7, no. 5.

6. ———, *Sopra la configurazione del pentaedro*, Rendiconti del Circolo Matematico di Palermo, vol. 21 (1906), pp. 322–341.

7. ———, *Una interpretazione geometrica del gruppo totale di sostituzioni sopra sei elementi*, Annali di Matematica Pura ed Applicata, 1909.

8. Corrado Segre, *Sulla varietà cubica con dieci punti doppi dello spazio a quattro dimensioni*, Atti della Accademia delle Scienze di Torino, vol. 22 (1886–1887), p. 791.

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# CORRECTION TO "ON GREEN'S FUNCTIONS IN THE THEORY OF HEAT CONDUCTION IN SPHERICAL COORDINATES"*

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In the article entitled *On Green's functions in the theory of heat conduction* by H. S. Carslaw and J. C. Jaeger (this Bulletin, vol. 45 (1939), pp. 407–413), a misprint is noted in the expression for $G$ on page 133 of my article *On the operational determination of two dimensional Green's functions in the theory of heat conduction* (this Bulletin, vol. 44 (1938), pp. 125–133), the correct expression for $G$ being

$$G = u + v = \frac{1}{4\pi} \sum_{n=-\infty}^{\infty} \cos n(\theta - \theta_0) \int_{-\infty}^{\infty} \alpha e^{-k\alpha^2 t} \frac{H_n^{(1)}(\alpha r_0)}{U_n(a\alpha)} \cdot \left\{ J_n(\alpha r) U_n(\alpha a) - H_n^{(1)}(\alpha r) \left[ \alpha \frac{d}{dz} J_n(z) + h J_n(z) \right]_{z=\alpha a} \right\} d\alpha,$$

where

$$U_n(\alpha a) = \left[ \alpha \frac{d}{dz} H_n^{(1)}(z) + h H_n^{(1)}(z) \right]_{z=\alpha a}.$$

When this correct expression is employed, formula (20), page 313, of the present paper becomes

$$\begin{aligned} G(r, \theta, \phi, t; r_0, \theta_0, \phi_0) &= \frac{1}{8\pi (r r_0)^{1/2}} \sum_{n=0}^{\infty} (2n + 1) P_n(\cos \gamma) \\ \text{(A)} \qquad &\cdot \int_{-\infty}^{\infty} \alpha e^{-k\alpha^2 t} \frac{H_{n+1/2}^{(1)}(\alpha r_0)}{U_{n+1/2}(\alpha a)} \left\{ J_{n+1/2}(\alpha r) U_{n+1/2}(\alpha a) - H_{n+1/2}^{(1)}(\alpha r) \right. \\ &\left. \cdot \left[ \alpha \frac{d}{dz} J_{n+1/2}(z) + (h - 1/(2a)) J_{n+1/2}(z) \right]_{z=\alpha a} \right\} d\alpha. \end{aligned}$$

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* This Bulletin, vol. 45 (1939), pp. 310–315.