A THEOREM ON CONTINUOUS FUNCTIONS IN ABSTRACT SPACES[1]

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In this note the following Theorem A concerning continuous functions in a very general abstract space is established, and from this theorem are deduced certain results concerning semi-metric spaces. In particular, Theorems 2.2 and 2.3 below generalize a theorem proved by Montgomery[2] concerning the behavior of the distances between points of a metric space under transformations of the space into itself.

The space S to be considered is a collection of "points" such that to each nonvacuous subset M of S there is defined a unique nonvacuous enclosure set $\overline{M}$ such that if M_0 and M_1 are nonvacuous subsets then $(\overline{M_0+M_1}) = \overline{M}_0+\overline{M}_1$. We shall not even require that M be a subset of $\overline{M}$, although unquestionably this latter condition is desirable for a far-reaching topological study of abstract spaces in general. Two sets M and N are *mutually separated* if $N\overline{M}+\overline{N}M=0$. A point set X is said to be *connected* if it is not the sum of two nonvacuous mutually separated point sets. Let $\phi(p)$ be a single-real-valued function defined on S. If M is a nonvacuous subset of S, we shall denote by $\phi(M)$ the set of real numbers $\phi(p)$ determined as p ranges over M. Using the usual absolute value as the metric in the space of real numbers, and defining the enclosure of a set of real numbers as the set, together with all its limit points, the function $\phi(p)$ will be said to be *continuous* on S if, for every subset M of $S, \phi(\overline{M})$ is contained in $\overline{\phi(M)}$.

THEOREM A. *Suppose S is a space of the above described sort which is connected, and $f(p, q)$ is a single-real-valued function defined for each pair (p, q) of S and such that:* (i) $f(p, p)=0$; (ii) $f(p, q)=f(q, p)$; (iii) *$f(p, q)$ is continuous in its arguments separately on S. Then either $f(p, q)$ is of constant sign (that is, for all (p, q) either $f(p, q)\leqq 0$ or $f(p, q)\geqq 0$), or there exists a pair of points $p\neq q$ such that $f(p, q)=0$.*

1. **Proof of Theorem A.** The conclusion of this theorem will be established by indirect argument. For suppose that this conclusion is not true. Then $f(p, q)\neq 0$ for $p\neq q$, and there exist points p_1, q_1, p_2, q_2

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[2] D. Montgomery, *A metrical property of point-set transformations*, this Bulletin, vol. 40 (1934), pp. 620–624.

such that $f(p_1, q_1) > 0$, $f(p_2, q_2) < 0$. It will now be proved that this condition is in contradiction to the hypotheses of the theorem.

We shall first show that this latter condition implies that there exist points x, y_1, y_2 such that $f(x, y_1) > 0$, $f(x, y_2) < 0$. If $q_2 = q_1$, we may take $x = q_1$, $y_1 = p_1$, $y_2 = p_2$; if $q_2 = p_1$ we may take $x = p_1$, $y_1 = q_1$, $y_2 = p_2$. There remains the case $q_2 \neq q_1$ and $q_2 \neq p_1$; in this case if $f(p_1, q_2) > 0$, take $x = q_2$, $y_1 = p_1$, $y_2 = p_2$, whereas, if $f(p_1, q_2) < 0$, take $x = p_1$, $y_1 = q_1$, $y_2 = q_2$. It is to be remarked that this part of the proof does not utilize the connectedness of S. The original proof of the author of this result involved the use of connectedness. The above simple proof was suggested by L. M. Graves.

Now for a point x let A_x denote the set of all points p such that $f(x, p) < 0$, and B_x the set of all points p such that $f(x, p) > 0$. On the assumption that the theorem is false it has been shown that there exists an x such that neither A_x nor B_x is vacuous. For each such x we have $x + A_x + B_x = S$. From the continuity of $f(x, p)$ as a function of p it follows readily that A_x and B_x are mutually separated; moreover, since $f(x, x) = 0$, $f(x, p) \neq 0$ for $p \neq x$, we have that

$$\bar{x} = x, \qquad \overline{(x + A_x)} \cdot B_x = 0, \qquad \overline{(x + B_x)} \cdot A_x = 0.$$

Finally, the connectedness of S implies $x \cdot \bar{A}_x = x$, $x \cdot \bar{B}_x = x$, and that $x + A_x$ and $x + B_x$ are connected sets.

Now consider an arbitrary point q of A_x. Then x is a point of A_q since $f(q, x) = f(x, q) < 0$. Suppose $B_q \cdot B_x \neq 0$. Then $q + B_q = (q + B_q) \cdot A_x + (q + B_q) \cdot B_x$, and the sets $(q + B_q) \cdot A_x$, $(q + B_q) \cdot B_x$ are mutually separated since A_x and B_x are mutually separated. But by the above argument the set $q + B_q$ is connected. Therefore, $B_q \cdot B_x = 0$, and B_x is a subset of A_q. Since q was an arbitrary point of A_x, we have for each q of A_x and each p of B_x that $f(q, p) < 0$. For p a fixed point of B_x, it follows from the fact that $x \cdot \bar{A}_x = x$ and the continuity of $f(q, p)$ as a function of q that $f(x, p) \leqq 0$. This, however, is impossible in view of the definition of B_x. Hence Theorem A is established.

2. **Applications of Theorem A.** We shall now consider a space S which is topologized by means of a real symmetric distance function $d(p, q)$. That is, $d(p, q)$ is defined for each pair of points (p, q) of S and: (a) $d(p, p) = 0$; (b) $d(p, q) = d(q, p)$; (c) $d(p, q) \neq 0$ if $p \neq q$. If M is a subset of S, the enclosure of M shall be defined as the set of all points p satisfying the condition that there is a corresponding sequence $\{p_n\}$ belonging to M such that $\lim_{n \to \infty} |d(p, p_n)| = 0$. The enclosure function for such a space is clearly additive, and hence such a space is of the kind considered above. For a space of this latter sort it is readily

seen that a subset M is contained in its enclosure $\overline{M}$; moreover, if M consists of a single point p, then $\overline{M}=\bar{p}=p$. Such a space is a semi-metric space in the sense of Fréchet if in addition $d(p, q) \geqq 0$ for all points p, q. Clearly a space S which is topologized by means of a real symmetric distance function is a semi-metric space with respect to the new distance function $|d(p, q)|$. In view of the above Theorem A we have, however, the further result:

THEOREM 2.1. *Suppose S is topologized by means of a real symmetric distance function $d(p, q)$. If S is connected and $d(p, q)$ is continuous in its arguments separately on S, then either $d(p, q)>0$ or $d(p, q)<0$ for all distinct points p and q; that is, either S is a semi-metric space with respect to $d(p, q)$, or S is a semi-metric space with respect to the distance function $-d(p, q)$.*

In view of this theorem, there is no loss of generality in stating the following result for a semi-metric space instead of for a space topologized by means of a real symmetric distance function.

THEOREM 2.2. *Suppose S is a connected semi-metric space whose distance function $d(p, q)$ is continuous in its arguments separately. If T is a continuous transformation on S to S, and there exist pairs of points (p_1, q_1), (p_2, q_2) such that $d(Tp_1, Tq_1)>d(p_1, q_1)$, $d(Tp_2, Tq_2)<d(p_2, q_2)$, then there exists a point pair (p,q), $(p\neq q)$, such that $d(Tp, Tq)=d(p,q)$.*

It is to be emphasized that we do not require that the image of S under T be the whole of S; moreover, if S_1 is the image of S under T, it is not supposed that there is a one-to-one correspondence between the points of S and S_1.

If $f(p, q)$ is defined as $d(Tp, Tq)-d(p, q)$, Theorem 2.2 is an immediate consequence of Theorem A. This latter theorem extends in several directions a result proved by Montgomery (loc. cit.). In particular, Montgomery assumed the space which he considered to be arc-wise connected.

In the above cited paper, Montgomery proved as a preliminary result that if S is a conditionally compact metric space and T is any one-to-one transformation of S into the whole of itself which does not leave all distances invariant, then T must increase at least one distance and decrease at least one distance. Actually, his proof applies to a semi-metric space whose distance function is continuous in the arguments (p, q) jointly. Morover, for his proof of an auxiliary lemma to be valid it is not necessary to assume that S itself is conditionally compact; it is sufficient to assume that if p is an arbitrary point of S, then the subset of S consisting of the points $[p, Tp, T^2p, \cdots]$ is conditionally compact. For convenience, we shall term such a transformation a *conditionally compact transformation.* If S itself is conditionally compact, then clearly every one-to-one transformation of S into itself is conditionally compact; it is readily seen, however, that there are non-trivial examples of conditionally compact transformations of a space S into itself for which the given space S is as a whole not conditionally compact. Montgomery's proof of the lemma in the above cited paper establishes the following result: *Suppose S is a semi-metric space whose distance function $d(p, q)$ is continuous in (p, q) jointly, and that T is a one-to-one conditionally compact transformation of S into the whole of itself. Then if T increases the distance between some two points of S, there also exist two points whose distance is decreased under T.* Hence we have the following conclusion:

LEMMA. *Suppose S is a semi-metric space whose distance function $d(p, q)$ is continuous in (p, q) jointly. If T is a one-to-one transformation of S into the whole of itself such that T and its inverse T^{-1} are each conditionally compact transformations, and T does not leave all distances invariant, then T must increase at least one distance and decrease at least one distance.*

The following result is then a consequence of this lemma and Theorem 2.2.

THEOREM 2.3. *Suppose S is a connected semi-metric space whose distance function $d(p, q)$ is continuous in its arguments (p, q) jointly. If T is a continuous one-to-one transformation of S into the whole of itself such that T and its inverse T^{-1} are each conditionally compact transformations, then there are two distinct points of S whose distance is invariant under T.*

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