ON INCIDENCE GEOMETRY

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The purpose of this note is to analyze the conditions needed in geometry to introduce ideal points without using order relations. Since only incidence relations are used, it is convenient to use the notation of lattice theory. The actual introduction of the ideal elements is a purely algebraic process belonging to the theory of ideal extension and will be given elsewhere. (See abstracts 44-5-201, 45-1-16, 45-1-17.) Beside conditions already familiar in lattice theory we need conditions for the existence of products (see definition of σ-lattice) and the obviously necessary condition for projectivization, Condition E. The conditions for the existence of products are needed because incidence geometry is not taken to be a lattice; in obtaining the projective extension it would be inconvenient to have to redefine the product of, say, two parallel lines; such a product, therefore, is left undefined. Condition E is not proved independent since our purpose is merely the elimination of considerations of order. Condition E has more force the greater the dimension of the space in which it operates. For dimension greater than 3 the development is consequently straightforward, so that we consider this case first. For dimension 3, however, Condition E appears to be a little too weak and we have a degenerate case requiring the use of the various forms of Desargues' theorem; the proof of these (D and D′) requires an axiom on the existence of transversals, Axiom T. The three-dimensional case is put last, but in it the connection with the classical theory (see Pasch-Dehn, Whitehead, and Baker) is most apparent.

1. **Geometric partial orderings.** For the case of any dimension greater than 2, we begin with "linear element" and $\leqq$ as undefined; the linear elements will later be classified according to their dimensions; also taken as undefined are the operations of joining: $a+b$, and intersecting: ab. For projective geometry these may be defined in terms of $\leqq$, but in general incidence geometry we wish to permit certain products ab to remain undefined for convenience in deriving extensions, hence axioms are added. "$\leqq$" is to be read as "on."

The following are the axioms for PARTIAL ORDERING:

1. $a \leqq a$ for every a.
2. $a \leqq b$ and $b \leqq c$ imply $a \leqq c$.

DEFINITIONS. $a=b$ *if* $a \leqq b$ *and* $b \leqq a$; $a<b$ *if* $\cdots$.

3. If A is a set of elements and $\sum(A)$ exists, then $a \leqq \sum(A)$ for all $a \varepsilon A$, and, if $a \leqq x$ for all $a \varepsilon A$, then $\sum(A) \leqq x$.

3′. If A is a set of elements and $\prod(A)$ exists, then $a \geqq \prod(A)$ for all $a \varepsilon A$, and, if $a \geqq x$ for all $a \varepsilon A$, then $\prod(A) \geqq x$.

4. $a \leqq b$ implies existence of $a+b$ (hence $=b$).

4′. $a \geqq b$ implies existence of ab (hence $=b$).

5. There is a $0 \leqq$ every a.

5′. There is a $1 \geqq$ every a.

MODULARITY. *If $c \geqq a$ and bc, $a+bc$, $a+b$, $c(a+b)$ exist, then $c(a+b)=a+bc$.*

This axiom leads to the classification of elements according to dimension.

ARCHIMEDEAN. *Every "chain" $a_1 > a_2 > \cdots$ or $a_1 < a_2 < \cdots$ is finite.*

DEFINITION. *a is "prime" over b if $a > b$ and no x exists with $a > x > b$. An element prime over 0 (under 1) is called a point (hyperplane). The Archimedean axiom assures the existence of both. The letter $p(h)$ is used to represent a point (hyperplane). The letter l is reserved for elements prime over a point (namely, lines).*

By a σ-lattice we mean an Archimedean, modular, partially-ordered set in which

1. if p is not on a, then pa exists (hence $=0$),
2. if $p \leqq a$, b, then ab exists,
3. $a+b$ always exists.

The essence of this definition is that the elements above a point form a modular lattice (that is, $a+b$ and ab always exist). Together with the complementation and irreducibility axioms given below, it implies that the elements above a point form a projective geometry.

In a σ-lattice one defines a "principal" chain as an ascending chain in which each element is prime over the preceding, the number of elements being called its length. It is then easy to show that the lengths of any two principal chains between a and b, where $a < b$, are equal. We then define dim a as the length of a principal chain between 0 and a minus 1, so that dim $0=-1$, dim $p=0$, and so on. Furthermore, $a < b$ implies dim $a <$ dim b, and, if ab exists, dim a $+$dim $b=$dim $ab+$dim $(a+b)$. To prove that $x=y$ by "counting dimensions" we show that $x \leqq y$ and dim $x=$dim y, usually by means of the above dimensional identity.

The following conditions are equivalent in any σ-lattice, so that any one may be used as an axiom, the property involved being called "complementation."

1. If $x \leqq b$, there is a y with $x+y=b$, and $\dim b = 1+\dim x+\dim y$.
2. If $a<b$, then $b=a+p_1+\cdots+p_r$, where p_i is not on $a+p_1+\cdots+p_{i-1}+p_{i+1}+\cdots+p_r$, and $\dim b=1+\dim a+\dim (p_1+\cdots+p_r)$.
3. If $0<b$, then $b=p_1+\cdots+p_r$, where p_i is not on $p_1+\cdots+p_{i-1}+p_{i+1}+\cdots+p_r$.
4. If $0<a$, then a is a sum of points.
5. If $a<b$, then there is a $p<b$ with p not on a (alternatively, if $p<b$ implies $p<a$, then $b \leqq a$).
6. If $a \leqq x \leqq b$, then there is a y with $a \leqq y \leqq b$, $x+y=b$, and $\dim a+\dim b=\dim x+\dim y$.

The condition of complementation gives the theory of linear dependence.

If L is a complemented σ-lattice, then $a+b \neq a, b$ implies that $1+\max\{\dim a, \dim b\} \leqq \dim (a+b) \leqq 1+\dim a+\dim b$.

With this in mind, we define, for the *general* σ-lattice: x and y are skew if $\dim (x+y)=1+\dim x+\dim y$; x and y are fully transversal if $\dim (x+y)=1+\max\{\dim x, \dim y\}$; the class $\{a_1, a_2, \cdots\}$ is called equi-transversal, written $E[a_1, a_2, \cdots]$ if any two are fully transversal and have the same dimension ($\dim (a_i+a_j)=1+\dim a_i$). This generalizes the relation of "coplanarity in pairs" used in §3. An equi-transversal class is called "maximal" if it contains an element on any given point. The ideal point in descriptive geometry was a maximal equi-transversal class; although planes, and so on, are added in the algebraic method, the maximal equi-transversal class is still the "backbone" of the ideal element.

In any σ-lattice, if $\dim a_1=\dim a_2$ and p is not on a_1+a_2, then $E[a, a_1]$, $E[a, a_2]$, $p \leqq a$ if and only if $a=(p+a_1)(p+a_2)$; and if $p_3<a_3<a_1+a_2$, $\dim a_3=\dim a_1$, then $E[a, a_3]$ if and only if $a_3=(p_3+a)(a_1+a_2)$. It consequently follows that if $E[a_1, a_2]$ and $a_1+a_2<1$, there cannot be more than one maximal equi-transversal class containing them, any two of its elements will determine it, and if a_1a_2 exists, the elements are given by $p+a_1a_2$ for all p not on a_1a_2.

To prove the existence of such maximal equi-transversal classes is a harder problem which will be considered in §2.

Consider now an Archimedean lattice for which

1. if p is not on a, then $p+a$ is prime over a;
2. if $a>ah>0$, then a is prime over ah;
3. if it is not true that $a \leqq b$, then there is a $p \leqq a$ with p not on b.

It is not difficult to show that such a lattice is complemented, so that Menger's axioms hold. We therefore call it a Menger lattice. It is also not very difficult to show that the modularity condition holds in the two cases (1) $bc>0$, (2) either $b\leqq c$ or $b\geqq c$. From this it follows that a complemented σ-lattice becomes a Menger lattice when the undefined products are defined to be 0, and, conversely, a Menger lattice in which those products are undefined for which $\dim a+\dim b>\dim(a+b)+\dim ab$ becomes a complemented σ-lattice.

A complemented σ-lattice is called irreducible if any line ($\dim=1$) contains at least three distinct points. The reason for this terminology, at least in the projective case, may be found in Menger, Birkhoff, and von Neumann. The primary use of this condition is in the proof that the projective extension is "minimal."

2. **Incidence geometry.** By an incidence geometry we mean an irreducible σ-lattice of dimension not less than 3 fulfilling the following condition:

CONDITION E. *If* $E[l_1, l_2, l_3]$, $E[l_1, l_2, l_4]$, *and* $l_1+l_2 \not> l_j$, *then* $E[l_3, l_4]$.

For dimension 3, the further condition T is required. See §3.

In an incidence geometry it is possible to prove the existence of a maximal equi-transversal class containing a_1 and a_2 if $E[a_1, a_2]$ and $a_1+a_2<1$; we have seen that the uniqueness holds in any σ-lattice. This result follows readily from three lemmas, the first two of which generalize Conditions E and E′ of §3.

LEMMA B_1. *If* $E[a_1, a_2, a_3]$, $E[a_1, a_2, a_4]$, *and if it is not true that* $a_j\leqq a_1+a_2$, *then* $E[a_3, a_4]$.

For, taking $p_i<a_i$, p_i not on $\prod a_i$, $a_1=\sum_{j=1}^{r} l_{1j}$ where $p_1<l_{1j}$, $r=\dim a_i$, we get $a_i=\sum_{j=1}^{r} l_{ij}$ where $l_{ij}=a_i(p_i+l_{1j})$, and $E[l_{1j}, l_{2j}, l_{3j}]$, $E[l_{1j}, l_{2j}, l_{4j}]$ (by a dimensional count), so that Condition E gives $E[l_{3j}, l_{4j}]$, and we then have $a_3+a_4=\sum l_{3j}+\sum l_{4j}=\sum(l_{3j}+l_{4j})=\sum(l_{3j}+p_4)=\sum l_{3j}+p_4=a_3+p_4$.

LEMMA B_2. *If* $E[x, x_i, x_j]$, $i, j=1, 2, 3$, $x_i+x_j=x_j+x_k$, *and* $x\leqq x_i+x_j$ *does not hold, and if* p' *is not on* x, x_i+x_j, *or* $x'=\prod(x_i+b')$, *then* $E[x, x', x_i]$.

For p' is on at most one $x+x_i$; hence we consider the following cases:

CASE 1. If $p'\leqq x+x_k$, let $x_i'=(x_i+p')(x+p')$; then $E[x, x_i, x_i']$ and $x_i'=(x+p')(x+x_k)$; thus $E[x_i', x, x_k, x_i]$, so that B_1 with $E[x, x_i, x_i']$, $E[x, x_i, x_j]$ gives $E[x_i, x_j, x_i']$; similarly $E[x_i, x_j, x_j']$ so that $x_i'=(x_i+p')(x_j+p')=x_j'$, therefore, $x_i'=x'=x_j'$, and so on.

CASE 2. If p' is not on $x+x_i$, $i=1, 2, 3$, let $x_i' = (x+p')(x_i+p')$ so that $E[x, x_i, x_i']$, and therefore B_1 and $E[x, x_i, x_j]$ give $E[x_i', x_j, x_i, x]$; therefore $x_i' = (x_i+p')(x_j+p') = x_j'$, therefore $x_i' = x' = x_j'$.

LEMMA B_3. *If* $E[x_1, x_2, x_3, x_4]$, $x_1+x_2 \not> x_i$, $x_3+x_4 \not> x_j$, *and* p *is not on* x_1+x_2, x_3+x_4, *then* $p \leqq x$ *has* $E[x_1, x_2, x]$ *if and only if* $E[x_3, x_4, x]$.

For $E[x_1, x_2, x_i]$ and B_1 give $E[x, x_i]$.

THEOREM B. *If* $E[x_1, x_2]$ *and* $x_1+x_2<1$, *then there is a* (*hence exactly one*) *maximal equi-transversal class containing* x_1 *and* x_2, *and* (1) *if* x, x', x'' *are in it*, $x'+x'' \not> x$, *then we have* $\dim (x+x'+x'') = 2+\dim x$; (2) *it is determined by any two of its elements*; *and* (3) *if* xx' *exists, then any element is on* xx', *for example*, $x''=p''+xx'$.

It is only necessary to show that the constructions in §1: $x=(p+x_1)(p+x_2)$, and $x_3=(p_3+x)(x_1+x_2)$ have x_3 independent of x and x independent of x_1, x_2. This is the substance of Lemmas B_2 and B_3.

THEOREM. *If* $h \not> x$, $\dim x < \dim h$, *then there is one and only one maximal equi-transversal class containing* x *with elements on* h.

For if $p'<h$, $x'=(x+p')h$, $p''<h$, p'' not on x', $x''=(x+p'')h$, then $E[x, x', x'']$ (by $x'+x'' \leqq (x+p'+p'')h$ and a dimensional count), so that the class determined by x' and x'' is the one sought.

DEFINITION. *If* p_1, p_2 *are not on* h *and* $x_1 \geqq p_1$ *with* $\dim x_1 < \dim h$, *then* $\Gamma^h_{p_1p_2}(x_1) = x_2$ *is the element on* p_2 *of the maximal E.T. class of* x_1 *and* h.

THEOREM Γ. 1. *If* x_1h *exists and* $x_1h>0$, *then the same holds for* x_2h *and* x_1x_2, *and* $x_1h=x_2h=x_1x_2$, $x_2=p_2+x_1h$.

2. *If* $x_1 \leqq y_1$, *then* $x_2 \leqq y_2$.

3. $\Gamma^h_{p_1p_2}(p_1)=p_2$, *and, if* $x_1 \geqq p_1+p_2$, $x_2=x_1$.

The proof follows readily from $x_2=[p_2+(x_1+p)h][p_2+(x_1+p')h]$, $y_2=[p_2+(y_1+p)h][p_2+(y_1+p')h]$ where $p, p'<h$ with $\dim (y_1+p+p')=2+\dim y_1$.

LEMMA. *If* $E[x_1, y_1, z_1]$ *and* $z_1 \leqq x_1+y_1$, *then* $E[x_2, y_2, z_2]$ *and* $z_2 \leqq x_2+y_2$.

By Γ this needs proof only if x_1+y_1 is a hyperplane. We must show that $p_2' \leqq z_2$ implies $p_2' \leqq x_2+y_2$; that is, $l_2=p_2+p_2' \leqq z_2$ implies $l_2 \leqq x_2+y_2$; take $p_1' \leqq x_1+y_1$ and not on l_1; then the lines l_1, $(l_1+p_1')x_1$, $(l_1+p_1')y_1$ correspond to $l_2<z_2$ and lines on x_2, y_2; if we can show that these coplane, then $l_2 \leqq x_2+y_2$; hence the lemma requires proofs for lines only, and therefore follows from Γ if $\dim 1>3$. If $\dim 1=3$, Baker's proof applies (using results of §1, this is the only point in our development in which Desargues' theorem is needed).

THEOREM. *$\Gamma^h_{p_1p_2}$ may be extended to hyperplanes in one and only one way if the conditions of Γ are to hold.*

We merely take $h_1=x_1+y_1$, $E[x_1, y_1]$, and define $h_2=x_2+y_2$; the lemma then applies.

Theorem F of §3 is an obvious consequence of the lemma, using $\Gamma^\pi_{p_1p_2}$; it generalizes to the following theorem.

THEOREM. *If $h=\sum_1^r l_{i1}=\sum_1^r l_{i2}$ with $E[l_{i1}, l_{i2}]$, $r=\dim 1$, and if p', p'' are not on h, $l_i'>p'$, $l_i''>p''$ belonging to the maximal E.T. class of l_{i1}, l_{i2}, then $E[\sum_1^r l_i', \sum_1^r l_i'']$.*

For $\Gamma^h_{p'p''}$ takes l_i' into l_i'', and hence $\sum l_i'$ into $\sum l_i''$.

Thus far, maximal E.T. classes are significant only if the elements are smaller than hyperplanes. Any class of hyperplanes is E.T. We can now distinguish the important ones.

DEFINITION. *A class of hyperplanes is called "regular" if there is one and only one on any p, and if, for every h, h_1, h_2 in it and $p_i<h_i$, we have $h_2=\Gamma^h_{p_1p_2}(h_1)$.*

THEOREM. *If $h\neq h_1$, there is one and only one R. class on them.*

The uniqueness is obvious. To prove existence, we must show that any h', h'' in the R. class determine it in the same way; that is, if $h_2=\Gamma^h_{p_1p_2}(h_1)$, $h'=\Gamma^h_{p_1p'}(h_1)$, $h''=\Gamma^h_{p_1p''}(h_1)$, then $h_2=\Gamma^{h'}_{p''p_2}(h'')$. We use the following lemmas to give $h_2=\Gamma^h_{p'p_2}h'$, $h''=\Gamma^h_{p'p''}h'$ (from $h_1=\Gamma^h_{p'p_1}h'$), hence $h_2=\Gamma^{h'}_{pp_2}h$, $h''=\Gamma^{h'}_{pp''}h$, therefore $h=\Gamma^{h'}_{p''p}h''$, and, finally, $h_2=\Gamma^{h'}_{p''p_2}h''$.

LEMMA 1. *If $x_2=\Gamma^h_{p_1p_2}(x_1)$ and $x_3=\Gamma^h_{p_2p_3}(x_2)$, then $x_3=\Gamma^h_{p_1p_3}(x_1)$.*

For x_1 and h give x_2, x_2 and h give x_3, hence x_1 and h give x_3; if $x_i=h_i$, split them up as usual.

LEMMA 2. *If $p<h$, $p_i<h_i$, and $h=\Gamma^{h_1}_{p_2p}(h_2)$, then $h=\Gamma^{h_2}_{p_1p}(h_1)$.*

For taking, as usual, $h_1=\sum_i^r l_{i1}$ with $p_1<l_{i1}$, $r=\dim h_1$, and $l_{i2}=h_2(p_2+l_{i1})$, we get $h_2=\sum_i^r l_{i1}$ and $l_{i1}=h_1(p_1+l_{i2})$; thus, on p, h_1 and l_{i2} give l_i; therefore l_{i1} and l_{i2} give l_i, hence l_{i1} and h_2 give l_i; but $\sum_i^r l_i=h$, which was to be proved.

The maximal E.T. classes and the R. classes give all the ideal elements except the ideal hyperplanes.

3. **Three dimensions.** This section will be devoted not only to filling in the gap mentioned above, but also to showing the relationship between our development and the classical work. In particular, in order to show that incidence geometry includes descriptive, projective, and affine geometry, we will start with the better known incidence axioms of geometry and show their relationship to our axioms.

We begin with the undefined terms "point," "line," and "plane" and the undefined relation "on," using the symbols p, l, π and $>$ or $<$ (depending upon the dimensionality). As axioms we take what are essentially the Hilbert incidence axioms:

(1) If $p<l$ and $l<\pi$, then $p<\pi$.

(2) If $p_1\neq p_2$, they are on one and only one line, called p_1+p_2.

(3) There is a p.

(4) For any p_1, there is a $p_2\neq p_1$.

(5) For any l, there is a p not on l.

(6) If p_3 is not on p_1+p_2, and $p_1\neq p_2$, the three points are on one and only one plane, called $p_1+p_2+p_3$.

From this the meaning of $p+l$ is clear.

(7) If $p<\pi_1$ and $p<\pi_2$ and $\pi_1\neq\pi_2$, then there is an $l>p$, with $l<\pi_1$, $l<\pi_2$; this l is called $\pi_1\pi_2$.

(8) Any line contains at least three distinct points.

(9) For any π, there is a p not on π.

The three space is defined similarly to $p+l$, and its uniqueness follows from Axiom (7); it will be designated by 1. We also use the null element 0. In general we define $a+b$ by using the maximum number of "independent" points which may be chosen from a and b. The product ab is not always defined, but we do take $ab=a$ if $a\leqq b$; $\pi_1\pi_2$ is defined in case the conditions of Axiom (7) hold; $l\pi=p$ if $\pi\not> l$, $p<l$, $p<\pi$; $l_1l_2=p$ if $l_1\neq l_2$, $p<l_i$, $pa=0$ if $a\not> p$; and $l_1l_2=0$ if $l_1+l_2=1$, that is, if l_i are "skew." It is easy to see that if l_1 and l_2 are skew and $p<l_1$, then $(p+l_2)l_1=p$.

The partial ordering axioms, Axioms 1–5′, are equivalent to Axiom 1 and the various definitions above. Modularity is easily verified by enumerating the six possibilities after the trivial cases (further $\leqq$ relations occurring) are eliminated, use being made of Axioms 2 and 6. The Archimedean axiom is obvious from the finite dimensionality. The axioms for a σ-lattice then follow from Axiom 7 above and the definitions following. The complementation condition follows from Axioms 2, 4, 5, 6, 9.

Beside the above axioms we also use the following one.

Axiom T. *If* $p < l_1, \cdots, l_n$ *and* l *is skew to each of the* l_i, *then there is a* $\pi > l$, $\pi \not> p$ *which is transversal to the* l_i, *that is*, $\pi l_i = p_i$.

This axiom is obvious in projective geometry and follows readily in descriptive geometry.

Definition. *The set of lines* $\{l_1, l_2, \cdots\}$ *is called equi-transversal if every two of them coplane. We write* $E[l_1, l_2, \cdots]$. *If there are more than two lines, we exclude the case where all lie in the same plane.*

In the development of ideal elements from descriptive geometry (see Pasch, Whitehead, and Baker), the following incidence theorems are proved.

Theorem D. (Desargues' theorem on a point.) *If* $p < l_i, l_i'$, $=(i=1, 2, 3)$, *and* $l_{ij} = (l_i + l_j)(l_i' + l_j') = l_{ji}$, $l^{ij} = (l_i + l_i')(l_j + l_j') = l^{ji}$, *then* l_{ij} *coplane if and only if* $l^{12} = l^{23} = l^{31}$.

Theorem D′. (Desargues' theorem in a plane.) *If* $p_1 + p_2 + p_3 = p_1' + p_2' + p_3'$ *is a plane, and* $l_i'' = p_i + p_i'$, $l_{ij} = p_i + p_j = l_{ji}$, $l_{ij}' = p_i' + p_j' = l_{ji}'$, $l_{ij} l_{ij}' = p_{ij}$, $l_i'' l_j'' = p^{ij}$, *then* p_{ij} *colline if and only if* $p^{12} = p^{23} = p^{31}$.

Theorem E. *If* $E[l_1, l_2, l_3]$, $E[l_1, l_2, l_4]$, *and* $l_1 + l_2 \not> l_j$, *then* $E[l_3, l_4]$.

Theorem E′. *If* $\sum_1^3 l_i$ *is a plane*, p_4, p_5 *are not on* $\sum_1^3 l_i$, *and* $\prod_1^3 (l_i + p_4) = l_4$, *then* $\prod_1^3 (l_i + p_5) = l_5$ *and* $E[l_4, l_5]$.

Theorem F. *If* $l_i, l_i' < \pi$, $(i=1, 2, 3)$, *and* p^1, p^2 *are not on* π, *then* $l_i' = (l_i + p^1)(l_i' + p^1)$ *coplane if and only if* $l_i^2 = (l_i + p^2)(l_i' + p^2)$ *do.*

In this section we will establish those relationships among these theorems needed to show that all are consequences either of E or of E′, and will finally be led to the use of E (which is obviously a necessary condition for projectivization) as an axiom permitting the introduction of ideal elements. It is not our purpose here to consider whether it is independent of Axioms (1)–(9) and T. In projective geometry D and D′ are known to be true, E and E′ are obvious since all the lines concur, and F is obvious since the triangles l_i and l_i' are perspective from some line. Also E′ and F are the hypothesis and conclusion of a Desargues' theorem in the plane when the points p_{ij} and p^{ij} do not exist.

1. That D implies D′ is obvious upon projecting the configuration of D′ through a point not in the plane.

2. T and D′ imply D—obvious.

3. E$'$ implies E; for Axiom (8) may be used to find $p<l_1+l_2$ with p not on l_1, l_2, and Axiom (7) gives $(p+l_3)(l_1+l_2)=l$; taking $p_i<l_i$, we have $(p_3+l)(p_3+l_1)(p_3+l_2)=l_3$, hence, by E$'$, $(p_4+l)(p_4+l_1)(p_4+l_2)=l_4$, and $E[l_3, l_4]$.

4. E implies E$'$ is a special case of Lemma B_2, §2.

5. E implies D$'$; the proof parallels the classical construction with variations when certain intersections do not exist. Let $l=p_{23}+p_{31}$. We must show that $p_{12}<l$ if and only if $p^{31}<l_2''$. Take p_1 not on $\sum p_i$, $l_{31}''=p_1''+p_{31}$, $p_3''<l_{31}''$ with $p_3''\neq p_1''$, p_{31} (Axiom (8)). Thus $p_{31}=ll_{31}l_{31}'l_{31}''$. Let $l_i=p_i+p_i''$, $l_i'=p_i'+p_i''$ for $i=1$, 3, and take $l_{23}''=p_3''+p_{23}$; thus $p_{23}=ll_{23}l_{23}'l_{23}''$. Now take $l_2=(l_1+p_2)(l_3+p_2)>p_2$, so that $E[l_1, l_2, l_3]$ (for $l_i<p_i''+l_{31}$); take $l_{12}''=(l_1+l_2)(l_{31}''+l_{23}'')>p_1''$, $l_2'=(p_2'+l_{12}'')(p_2'+l_{23}'')>p_2'$, so that $E[l_2', l_{12}'', l_{23}'']$; and finally take $l'=(p^{31}+l_1)(p^{31}+l_3)>p^{31}$, so that $E[l', l_1, l_3]$.

Since $l_2<l_3+p_2=l_3+p_{23}$, $l_{23}''=p_3''+p_{23}<l_3+p_{23}$, the definition of l_{12}'' gives $E[l_2, l_{12}'', l_{23}'']$; since $l_i'<p_i''+l_{31}'$, $l'<p^{31}+l_i=p^{31}+p_i+p_i''=p^{31}+p_i'+p_i''=p^{31}+l_i'$, $l_2'<p_2'+l_{23}''=p_3'+l_{23}''>p_3'+p_3''=l_3$, we have $E[l', l_1', l_3']$ and $E[l_2', l_3']$. Consequently, applying E to $E[l', l_1, l_3]$, $E[l_2, l_1, l_3]$ and $E[l_2', l_{12}'', l_{23}'']$, $E[l_2, l_{12}'', l_{23}'']$, we get $E[l', l_1, l_2, l_3]$ and $E[l_2, l_2']$.

Now $p^{31}<l_2''$ if and only if $E[l', l_2']$; for since $E[l_2, l_2']$, $p^{31}<l_2''$ gives $l_2'<l_2+p_2'=l_2+l_2''=l_2+p^{31}=l_2+l'$, and, conversely, if $l_2''\not> p^{31}$ and $E[l', l_2']$, then $l_2+l_2'\not> l'$, so that $E[l', l_2', l_2]$; with $E[l_{23}'', l_2', l_2]$ and Theorem E, this gives $E[l', l_{23}'']$, so that we would have $p_{23}<l'+p_{23}=l'+l_{23}''=l'+p_3''=l'+l_3'=l'+p_3'=l'+l_3''$, and therefore $p_{23}<l_3''$, $l_{23}=l_3''=l_{23}'$, contradicting $l_{23}l_{23}'=p_{23}$. Thus we need only show that $E[l', l_2']$ if and only if $p_{12}<l$.

A. If $E[l_1, l_2']$, then $E[l', l_2', l_3']$ and $E[l', l_1', l_3']$ and Theorem E give $E[l_1', l_2', l_3']$; now if $l_1'+l_2'\not> l_{12}''$, we would have $E[l_{12}'', l_1', l_2']$ and therefore $p_1''=l_1'l_{12}''<l_2'$, hence $p_1''=l_1'l_2'<l_3'=p_3'+p_3''$, and therefore $p_3'<p_1''+p_3''=l_{31}''$ and $l_{31}''=p_{31}+p_3'=l_{31}'$, so that $p_1''<l_{31}'<\sum p_i$, a contradiction; hence $l_{12}''<l_1'+l_2'$, so that

$$\begin{aligned} p_{12}=l_{12}l_{12}' &= [(\textstyle\sum p_i)(l_1+l_2)][(\textstyle\sum p_i)(l_1'+l_2')] \\ &= (\textstyle\sum p_i)(l_1+l_2)(l_1'+l_2') \\ &< (l_1+l_2)(l_1'+l_2')=l_{12}'' \\ &< l_{31}''+l_{23}''; \; p_{12}<(\textstyle\sum p_i)(l_{31}''+l_{23}'')=l. \end{aligned}$$

B. If $p_{12}<l$, then $p_{12}=ll_{12}=[(\sum p_i)(l_{31}''+l_{23}'')][(\sum p_i)(l_1+l_2)]=(\sum p_i)(l_{31}''+l_{23}'')(l_1+l_2)<l_{12}''$; hence $l_{12}''+l_2'=l_{12}''+p_2'=l_{12}''+l_{12}'$,

so that $E[l_1', l_2']$, therefore $E[l_1', l_2', l_3']$, and Theorem E gives $E[l', l_2']$, which was to be proved.

6. D, E, and E′ imply F. The proof is the same as that in descriptive geometry, where use is made of the perspectivity of the elements on p^1 and p^2 with respect to π. It has already been indicated in §2.

In the development, E and E′ are used as criteria for the identity of ideal points and F is used to determine the collinearity of ideal points, whereas in descriptive geometry D and D′ are used to prove E′; if E is accepted they are needed only to prove F, and even so only in three dimensions. E′ may be used to define $E[l_1, l_2, \cdots]$ in the case where the l_i coplane.

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