ABSTRACTS OF PAPERS

SUBMITTED FOR PRESENTATION TO THE SOCIETY

The following papers have been submitted to the Secretary and the Associate Secretaries of the Society for presentation at meetings of the Society. They are numbered serially throughout this volume. Cross references to them in the reports of the meetings will give the number of this volume, the number of this issue, and the serial number of the abstract.

138. G. E. Albert: *On contiguous point spaces and their applications.*

In topologizing the class P^* of cyclic elements of a Peano space P, R. L. Moore (Rice Institute Pamphlet, vol. 23, no. 1) was led to the concept of contiguity spaces. The presence of pairs of contiguous points seems to make the theory of Moore's spaces quite complicated. In a simpler axiomatic treatment of contiguity, Theodore Hailperin (this Bulletin, vol. 45 (1939), p. 172) modified Hausdorff's neighborhood space by denying the separation axiom and defining two points as contiguous if every neighborhood of each contains the other. The symmetry of this definition implies that the relation is transitive; this should render his theory inapplicable to the study of P^*. In the present paper a neighborhood space is introduced which differs from ordinary topological spaces in essentially one respect, namely, the Hausdorff separation axiom is replaced by the unsymmetrical Kolmogoroff axiom: of every two points, at least one has a neighborhood not containing the other. If every neighborhood of the point x contains the point $y \neq x$, then x is said to be contiguous to y. It is the purpose of the paper to construct, on this basis of contiguity, a simple and comprehensive theory of the class P^*. (Received December 11, 1939.)

139. Garrett Birkhoff: *On a class of positive matrices.*

The paper deals with matrices of non-negative elements, the sum of the terms in every row and column being the same. It is first shown combinatorially that every such matrix is a positive linear combination of permutation matrices. Applications are then made to the theory of dependent probabilities, where such matrices occur; in particular, an ergodic theorem over arbitrary semigroups is proved—valid also for an analogous class of linear (stochastic) operators on the space (L). (Received December 21, 1939.)

140. Nathaniel Coburn: *A characterization of Schouten's and Hayden's deformation methods.*

Another approach to deformation problems, in an n-dimensional space with a connection L_n, is given. The principal idea is that all deformations take place in the local tangent space E_n at each point of L_n. First, the laws of deformation of differential and ordinary vector fields are examined. These laws are distinct in non-Finsler space; they coincide in Finsler space. It is shown that: (1) Schouten's deformation formulas are obtained when the coordinate differential vectors and the unit affinor (A_μ^λ) of L_n are deformed by E_n parallelism; (2) Hayden's deformation formulas are obtained when the measure vectors of L_n are deformed by L_n parallelism, and the vectors of a subspace as well as the unit affinor of L_n are deformed by E_n parallelism. Finally, a particular type of deformation condition is examined. The following theorems are proved: (1) if this deformation is compatible with a motion in L_n, then the Christoffel symbols of L_n are not deformed; (2) if this deformation is possible, then L_n is generally a space of absolute parallelism; (3) subcases of (2) are discussed. (Received December 4, 1939.)

141. Nathaniel Coburn: *Conformal unitary spaces.*

In this paper the properties of unitary spaces of n dimensions K_n and $'K_n$ whose fundamental tensors are related by the conformal transformations are studied. First, the relation between conformal spaces and the geodesics of these spaces is discussed. The following theorems are proved: (1) two unitary spaces, both with symmetric connection (without torsion), cannot be conformal; (2) if two unitary spaces are conformal, then their geodesics correspond; (3) conversely, if two unitary spaces, one of which is without torsion, are in a *restricted* geodesic correspondence and if a certain curvature affinor $'C_{\dot{\alpha}\bar{\mu}\nu}{}^{\cdot\beta}$ vanishes, then the two spaces are conformal. The paper concludes with a study of some special conformal unitary spaces. After defining the terms: (1) unitary k-spaces; (2) k-spaces which are k-conformal, the following theorems are proved: (1) the affinor $'C_{\dot{\alpha}\dot{\bar{\mu}}\dot{\nu}}{}^{\beta}$ is a conformal invariant of all k-spaces which are k-conformal; (2) conversely, if two unitary spaces are conformal and the affinor $'C_{\dot{\alpha}\dot{\bar{\mu}}\dot{\nu}}{}^{\beta}$ is a conformal invariant, then the spaces are k-spaces. A special case of these k-spaces has been studied by Bergmann, Mitrochin, Fuchs and others. (Received December 4, 1939.)

142. W. C. Graustein: *Harmonic minimal surfaces.*

A minimal surface in a euclidean space of three dimensions is harmonic if it is representable in terms of Cartesian coordinates (x_1, x_2, x_3) by an equation of the form $U(x_1, x_2, x_3) = \text{const.}$, where U is a harmonic function. It is the purpose of this paper to determine all the families of harmonic minimal surfaces. The families found are a pencil of planes, a family of right helicoids, a family of imaginary transcendental surfaces, a family of imaginary quartic surfaces, a family of imaginary sextic surfaces (for which U depends on an elliptic integral), and families of imaginary cylinders with isotropic rulings. The method employed introduces three mutually orthogonal congruences of curves, with unit tangent vector fields α, β, γ, which are closely associated with the required family of surfaces, and expresses the prescribed properties of these surfaces by a suitable choice of the coefficients in the equations of variation of α, β, γ, with respect to the arcs of the curves of the three congruences. These equations of variation constitute the differential system finally to be integrated. Their conditions of integrability yield a second differential system of ten partial differential equations of the first order in five dependent and three independent variables. (Received December 7, 1939.)

143. D. W. Hall and G. T. Whyburn: *On arc-preserving and tree preserving transformations.*

In an earlier abstract (45-11-399) the authors have obtained results concerning arc-preserving transformations. In the present paper the following additions to the theory are obtained: (a) Condition (i) (a) of the earlier abstract characterizes tree preserving transformations. (b) If $T(A) = B$ be continuous, then in order that T be tree preserving it is necessary and sufficient that the image of every simple arc in A be a tree in B. (c) If B is cyclic, then the following types of transformations are equivalent: arc-preserving, tree preserving, A-set reversing, monotone retracting. The set A is, throughout the paper, a compact locally connected continuum. (Received December 27, 1939.)

144. P. R. Halmos: *Statistics, set functions, and spectra.*

The purpose of this paper is to exploit the analogy existing between the distribution functions of statistics on the one hand and spectral families of projection operators on the other hand. It is shown that the class of all bounded measurable functions in the unit interval satisfies the axiomatic system developed by von Neumann (Matematicheskii Sbornik, vol. 43 (1936)) for the class of all bounded Hermitian operators on Hilbert space. Certain well known theorems, such as the equality of the expectation of a chance variable with the first moment of its distribution function, and the Radon-Nikodym theorem concerning absolutely continuous set functions are special cases of the general theory considered. These theorems are suggested by the application of Hilbert space theorems to real function theory. Working in the converse direction, certain concepts (particularly independence) of real function theory are extended to operators and the behavior of sets of independent operators is investigated, yielding results parallel to the usual results of probability theory. (Received January 3, 1940.)

145. E. D. Hellinger: *Classes of monotone functions.* Preliminary report.

The problem is to investigate the manner of increase of monotone functions over an interval or over a perfect set of points and to form classes of functions with common behavior in this respect. A function $g(x)$ is in the class defined by $h(x)$ if it can be represented by an integral of the form $\int_a^x (df(x))^2/dh(x)$ with a suitable $f(x)$; the integral is of the kind introduced by the author as a generalization of the Stieltjes integral (T. H. Hildebrandt, this Bulletin, vol. 24 (1918), p. 194). By these and other related integrals conditions are found under which two functions shall belong to the same class; further representatives of the different classes are given. It is shown that the theory is closely connected with the theory of absolute continuity. The theory can be applied, for instance, to the problem of the orthogonal equivalence of infinite quadratic forms or symmetric operators, bounded as well as non-bounded, which is just the problem which suggested these investigations. (Received December 27, 1939.)

146. Henry Hurwitz: *Total regularity of infinite matrix transformations.*

Total regularity of a matrix transformation was defined by W. A. Hurwitz who found a necessary and sufficient condition for it in the case of a real infinite triangular matrix (this Bulletin, vol. 28 (1922), p. 30). In the present paper a necessary and sufficient condition is deduced for a real regular infinite square matrix transformation given by $X = \lim \sum_{k=0}^{\infty} a_k(t)x_k$. There must eventually be only a finite number of negative elements $a_k(t)$ for each t, and there must be no sequence t_i such that a suitably defined set of negative elements appearing in the sequence is not properly "guarded" by large positive elements which also appear. Several modifications of the condition are possible. (Received December 15, 1939.)

147. A. N. Lowan and Gertrude Blanch: *Analysis of computing errors in the process of analytic continuation.*

When the values of a function $f(x)$ are to be tabulated over a fairly wide range of the argument x, it is frequently convenient to compute the function and its derivatives for certain equally spaced "key arguments" and to sub-tabulate by means of Taylor expansions around the key arguments. (It is assumed that all required derivatives are continuous over the entire range.) The scheme adopted by the "Project for the Computation of Mathematical Tables," W.P.A., New York City, involves computing in succession derivatives of all orders needed for every point in the range, beginning with those of the highest order, and ending with the function itself, keeping the interval constant. No estimate of the manner in which the error in the method is propagated from stage to stage seems to be available in the literature, and because of the importance of the method as a means of computation, an upper bound of the error is derived in this paper. (Received December 20, 1939.)

148. A. V. Martin and J. H. Roberts: *Two-to-one transformations of 2-manifolds.* Preliminary report.

If A is a compact 2-manifold, then a continuous two-to-one transformation can be defined on A if and only if the Euler characteristic $\lambda(A)$ is even. If B is the image space, then $\lambda(B)=\lambda(A)/2$. The possible image spaces include all the manifolds B (in number 0, 1, or 2) with $\lambda(B)=\lambda(A)/2$, as well as certain 2-complexes obtained by identifying pairs of points of a manifold. (Received December 29, 1939.)

149. Marston Morse and C. B. Tompkins: *Minimal surfaces of unstable type.*

The authors continue the theory initiated in the April issue of the Annals of Mathematics extending it to surfaces bounded by several contours. A typical result extending a classical result well known for surfaces of revolution bounded by two circles and generated by a catenary is as follows. Let g_1 and g_2 be two simple rectifiable closed curves whose convex envelopes do not intersect. Suppose that the ratio of chord to arc on g_i, $(i=1, 2)$, is bounded from zero. Term a surface S a ring-surface or disc-surface if S is the continuous image of a circular ring or a circular disc, respectively. Referring the term "minimizing" to the Douglas-Dirichlet integral, our theorem states the following. If g_1 and g_2 bound a ring mimimal surface of minimizing type (relative or absolute), then g_1 and g_2 also bound a ring minimal surface not of minimizing type, or else at least one of the two contours g_i bounds a disc minimal surface of unstable (non-minimizing) type. This result is easily verified for the classical case of surfaces of revolution, where the ring surface of unstable type always exists under the hypotheses. (Received December 1, 1939.)

150. Tibor Radó: *On a lemma of McShane.*

In his work on the semi-continuity of double integrals in the calculus of variations, McShane (Annals of Mathematics, (2), vol. 34 (1933), pp. 829–830, Lemma 7) established an important lemma which plays a fundamental part in his proofs. Using the notations of that paper, the main result of the present paper can be described by the following statement. The conclusion of the lemma of McShane remains valid if the functions x_0^i are subjected to the (obviously necessary) condition that the Jacobians X_0^i be summable. The proof is based on methods developed by the author in his work on the area of surfaces. Applications will be considered on another occasion. (Received December 13, 1939.)

151. A. L. Whiteman: *Additive prime number theory in real quadratic fields.*

In a series of three papers (Abhandlungen aus dem Mathematischen Seminar der

Hamburgischen Universität, vol. 3 (1924), pp. 109–163 and pp. 331–378; Mathematische Zeitschrift, vol. 27 (1926), pp. 321–426) Rademacher considered the problem of representing numbers in an algebraic field as the sum of primes. By applying the Hardy-Littlewood method he showed that if a certain hypothesis concerning the distribution of the zeros of Hecke's $\zeta(s, \lambda)$-functions is true, then every sufficiently "large" "odd" number in an algebraic field is the sum of three primes. In this paper it is proved that the same hypothesis implies that "almost" every "even" integer in a real quadratic field is the sum of two primes. The proof makes use of the Hardy-Littlewood-Rademacher method and also employs certain simplifications due to Landau. (Received December 22, 1939.)

152. R. P. Agnew: *Tauberian conditions.*

Let $K>0$ and $0<\theta<\pi/2$. Corresponding to each angle ψ, let $S(K, \theta, \psi)$ denote the "sector with vertical angle less than π" consisting of all points z of the complex plane representable in the form $z=-Ke^{i\psi}+\rho e^{i(\psi+\phi)}$ where $\rho\geqq 0$, $-\theta\leqq\phi\leqq\theta$. A series $\sum u_n$ of complex terms is said to satisfy the Tauberian condition T if K, θ, λ, and $\psi_1, \psi_2, \psi_3, \cdots$ exist such that $K>0$, $0<\theta<\pi/2$, $\lambda>1$, and, for each sufficiently great index k, $nu_n \varepsilon S(K, \theta, \psi_k)$ when $k\leqq n<\lambda k$. The condition T is sufficiently general to include classic Tauberian conditions of "order" and "gap" types, and hence provides the means of both generalizing and unifying classic Tauberian theorems. Characterizations and properties of the class of series satisfying T are developed. A condition T^* which is more general than T (and is related to T in much the same way that the condition that $\sum u_n$ be "langsam oszillierende" is related to the condition $n|u_n|<K$) is given, characterized, and discussed. Tauberian theorems involving T and T^* are proved for C_1 and other methods of summability. (Received January 10, 1940.)

153. R. P. Agnew and Mark Kac: *Translated functions and statistical independence.*

In connection with some problems of Kampé de Feriet concerning the theory of turbulence, Steinhaus proposed the following question: Does there exist a continuous function $f(t)$, $(-\infty<t<\infty)$, such that, for each sequence of real numbers $\lambda_1, \lambda_2, \cdots$, the functions $f(t+\lambda_1), f(t+\lambda_2), \cdots$ are statistically independent? (For the definition of statistical independence see M. Kac and H. Steinhaus: *Sur les fonctions indépendantes* IV, Studia Mathematica, vol. 7, pp. 1–15, and P. Hartman, E. R. van Kampen and A. Wintner: *Asymptotic distributions and statistical independence*, American Journal of Mathematics, vol. 61 (1939), pp. 477–487.) A positive answer to this question is given by the example $f(t)=\sin e^{t^2}$. A more general class of functions having the above property can be obtained. It may be mentioned that once the statistical independence is established, one can get different results concerning superposition of translations of the function. For instance, the relative measure of the set of those t's for which $f(t+\lambda_1)+\cdots+f(t+\lambda_n)<\alpha(n)^{1/2}$, $(f(t)=\sin e^{t^2})$, tends to $\pi^{-1/2}\int_{-\infty}^{\alpha}\exp(-u^2)du$ as $n\to\infty$. (Received January 13, 1940.)

154. G. E. Albert: *On quasi-metric spaces.* Preliminary report.

The present note studies a modified form of the quasi-metric spaces introduced by W. A. Wilson (American Journal of Mathematics, vol. 53 (1931), p. 675). Here, a space S will be called quasi-metric if for every pair of points x, y in S two non-negative numbers xy and yx can be defined such that (1) $xy=0=yx$ if and only if $x=y$, and (2) $xz\leqq xy+yz$ for every triple of points x, y, z in S. (Wilson used the axiom: $xy=0$ if and only if $x=y$.) If $xy=0$ but $yx\neq 0$, then x is called contiguous to y. Let $\{x_n\}$ be a sequence of points; one says that $\{x_n\}$ converges to x as an l-limit, r-limit, or c-limit accordingly as $xx_n \to 0$, $x_n x \to 0$, or both, respectively. One defines the corresponding classes of l-closed, r-closed, and c-closed sets as well as l-, r-, and c-open sets. The relationships between quasi-metric spaces and topological spaces are studied in detail. In particular, it is shown that the former include as special cases the contiguity spaces recently defined by the author (abstract 46-3-138) with neighborhood as the primitive concept. (Received January 30, 1940.)

155. Reinhold Baer: *Abelian groups that are direct summands of every containing abelian group.*

A characterization of those abelian groups, admitting a ring of operators, which are direct summands of every containing abelian group is given. It is shown that they are essentially a generalization of the groups "with division" and that they have similar properties. (Received January 5, 1940.)

156. A. C. Berry: *The accuracy of the Gaussian approximation to a sum of independent variates.*

The sum of finitely many variates frequently possesses an almost Gaussian distribution. Let M denote the least upper bound of the modulus of the difference between the distribution function of the variate sum and that of a related normal variate. M measures the "abnormality" of the variate sum. For uniformly bounded, totally independent variates, the following inequality is established: $M \leqq 2L/\sigma$. Here L is the least common bound of the *individual* variates, none of the given variates differs from its mean value by more than L except at most in cases of zero probability, and σ is the standard deviation of the variate *sum*. This result is an arithmetical refinement of the theorem of Liapounoff that $M \to 0$ when $L/\sigma \to 0$. The constant 2 is not the best possible. The existence of a best constant is proved, also that it cannot be less than $(2\pi)^{-1/2}$. The result rests on an equality connecting the difference of distribution functions with the difference of *characteristic* functions, an instance of the Parseval theorem in the theory of Fourier transforms. With the aid of a method of Paul Lévy, an extension is obtained for the case of unbounded variates. (Received January 25, 1940.)

157. W. Z. Birnbaum and H. S. Zuckerman: *On the properties of a collective.*

This paper is concerned with the concepts of a "collective" and a "selection" ("Stellenauswahl"), used in von Mises' foundations of the calculus of probabilities. The central theorem is: The set of all infinite selections can be interpreted as a space in which a Lebesgue measure is defined; then, if a sequence of 0's and 1's fulfills the first postulate of von Mises, it fulfills the second postulate in the sense that the limit of the relative frequencies is invariant for almost all selections in the space. (Received January 22, 1940.)

158. A. B. Brown: *On transformation of multiple integrals.*

The formula for transformation of a multiple Riemann integral under a change of variables $x_i = x_i(u_1, \cdots, u_n)$, $(i=1, \cdots, n)$, depends on the formula for the volume V in (x)-space of the image, under the transformation, of a solid R in (u)-space, namely $V = \int \cdots \int_R |J_{xu}| du_1 du_2 \cdots du_n$, where J_{xu} is the Jacobian. The standard proof of the latter formula is sufficiently forbidding, for $n \geqq 3$, to discourage one's presenting it to a class in advanced calculus. In the case $n=2$, the proof does not make it clear what formula will be obtained when n is greater than 2. A different method of proof gives the result quickly and naturally, but has not been given completely in the literature, to the writer's knowledge. In this note a simple but complete derivation of the formula is given following the second method. (Received January 17, 1940.)

159. L. E. Bush: *An asymptotic formula for the average sum of the digits of integers.*

Let $S(r, N)$ be the sum of the digits of all non-negative integers less than N, when these numbers are written in the scale of notation of radix r, so that $S(r, N)/N$ is the average sum of the digits of all numbers less than N (including zero) when these numbers are written in the r-scale. An asymptotic formula is found for $S(r, N)/N$ by which it is shown that for a sufficiently large N the average sum of the digits of all non-negative integers less than N is least when the numbers are written in the binary scale, and in general is smaller for a smaller radix than for a larger. (Received January 19, 1940.)

160. J. Hobart Bushey: *Expansions of products of certain symmetric orthogonal polynomials.*

In 1878 Adams gave (Proceedings of the Royal Society of London, vol. 27, pp. 63–71) a formula for the expansion of the product $P_m(x)P_n(x)$ of two Legendre polynomials in a series of Legendre polynomials, and made use of this expansion to evaluate $\int_{-1}^{1}P_m(x)P_n(x)P_r(x)dx$, $(m, n, r=0, 1, 2, \cdots)$. In the present paper, analogous results are obtained for symmetric Jacobi polynomials and for Hermite polynomials. (Received January 29, 1940.)

161. Sister M. Patricia Callaghan: *Generalized Frégier curves.*

The author finds equations of envelopes of generalized Frégier theorems. If in the Frégier theorem a constant angle is used instead of the right angle, the envelope is a conic. The equation of the envelope of this envelope is found as the point moves around the given conic. If through any fixed point on a conic, pairs of lines are drawn making supplementary angles with a fixed line in the plane, the envelope of the chords which join the intersections of these pairs of lines with the conic is, in general, a hyperbola. The equation of the envelope of this envelope is found as the fixed point moves around the given conic. (Received January 27, 1940.)

162. R. H. Cameron and W. T. Martin: *An unsymmetric Fubini theorem.*

Under fairly general conditions the authors prove that $\int s(u)d\int p(x, u)dk(x) = \int dk(x)\int s(u)dp(x, u)$, where the integrals are Lebesgue-Stieltjes (Radon) integrals and all the limits are from $-\infty$ to ∞. (Received January 12, 1940.)

163. Richard Courant and N. Davids: *Minimal surfaces spanning closed manifolds.*

The methods for the solution of the Plateau-Douglas problem must be considerably modified if "free boundaries" on prescribed boundary manifolds are considered. The present paper, extending previous results (Proceedings of the National Academy of Sciences, vol. 24 (1938), p. 97 ff., and Acta Mathematica, vol. 72 (1940)) solves the problem of constructing an extremal minimal surface whose boundary is free on a given closed manifold of genus $p>0$. The boundary is required to be linked with a prescribed cycle which itself is linked with the manifold. This novel type of topological conditions in the calculus of variations is essential for the formulation and solution of a wide class of problems concerning minimal surfaces. (Received January 17, 1940.)

164. M. M. Day: *Linear methods of summability.*

Let Y be a directed set; that is, a partially ordered set with the transitivity and composition properties of Moore and Smith (American Journal of Mathematics, vol. 44 (1922), pp. 102–121). Let V be a space of real-valued functions on Y such that $\lim_y f(y)$ exists in the Moore-Smith sense if $f \varepsilon V$. Let X be another directed set and for each $x \varepsilon X$ let U_x be a functional on V; the transformation thus defined is regular on V if $\lim_x U_x(f)=\lim_y f(y)$ for all $f \varepsilon V$. Extensions of the theorem of Toeplitz for simple sequences are shown to hold when V is a Banach space and the set X is essentially sequential in the sense that there exists a sequence $\{x_n\} \subset X$ such that each $x \varepsilon X$ is followed by some x_n. The weak topologies in the conjugate space of V and the descriptive theory of directed sets, due to Tukey (Thesis, Princeton University, 1939), are used in showing that some such restriction is necessary. The results are applied to finding regularity conditions for various function classes defined on particular directed sets, such as multiple sequences, functions of n real variables, continuous functions, and unconditionally convergent series. (Received January 26, 1940.)

165. M. M. Day: *The spaces L^p with $0<p<1$.*

The class $L_\mu^p(Y)$ is the set of all real-valued functions f defined on Y, measurable μ and such that $\int_Y |f|^p d\mu<\infty$, where μ is a measure (Saks, *Theory of the Integral*, chap. 1) and $0<p<1$. It is shown that a non-identically-zero, additive, and continuous functional on $L_\mu^p(Y)$ exists if and only if there is a measurable set E with $0<\mu(E)<\infty$, such that E cannot be divided into two disjoint sets of positive measure. The proof proceeds stepwise, first assuming $\mu(Y)<\infty$, next that Y can be split into a countable sum of sets of finite measure, and finally removing all such restrictions. As a corollary, no such nonzero functionals exist when Y is any measurable subset of a euclidean n-space and μ is Lebesgue measure. In the case when they exist, a general form of the linear functionals is given, and the class of them is shown to be equivalent to the class of bounded functions on a properly chosen set. (Received January 22, 1940.)

166. R. P. Dilworth: *A characterization of lattices of ideals.*

An element a of a continuous lattice is said to be finite dependent if $a \supset \Pi(S)$ implies $a \supset \Pi(S')$ where S' is a finite subset of S. The following theorem is proved: A continuous lattice $\mathfrak{S}$ is isomorphic to the lattice of ideals of a sublattice if and only if the finite dependent elements are closed with respect to union and generate $\mathfrak{S}$ under infinite crosscut. It follows that every exchange lattice is a lattice of dual ideals. (Received January 29, 1940.)

167. F. G. Dressel: *A Stieltjes integral equation.*

The paper shows how to solve a particular type of Young-Stieltjes integral equation. Also it is shown that only one of the three conditions imposed by Fischer (Annals of Mathematics, (2), vol. 25 (1923–1924), pp. 142–158) on the kernel of a Stieltjes integral equation is needed to insure the solution of this equation. (Received January 24, 1940.)

168. F. G. Dressel: *The fundamental solution of the parabolic equation.*

The existence of the fundamental solution for the parabolic equation with variable coefficients is proved in this paper. (Received January 24, 1940.)

169. W. K. Feller: *On the integrodifferential equations of the purely discontinuous Markoff processes.*

Consider a random variable $X(t)$ varying in some space E so that if $X(t)$ coincides with the point x the probability that no change will occur during $(t, t+\Delta t)$ is $p(t, x)\cdot\Delta t+o(\Delta t)$, and the probability of $X(t+\Delta t)$ being contained in the set $\Lambda \not\ni x$ is $p(t, x)\Pi(t, x, \Lambda)\cdot\Delta t+o(\Delta t)$, where $\Pi(t, x, \Lambda)$ is a probability distribution. It is examined to what extent these conditions determine the transition probabilities $P(\tau, x; t, \Lambda)$ of some stochastic process. Generalizing previous results (*Zur Theorie der stochastischen Prozesse,* Mathematische Annalen, vol. 113 (1936), and W. Dubrovski, *Eine Verallgemeinerung der Theorie der rein unstetigen stochastischen Prozesse von W. Feller,* Comptes Rendus de l'Académie des Sciences de l'URSS, vol. 19 (1938)) it is shown that, if all functions depend continuously on t, a derivative $\partial P/\partial t$ exists for a suitable class of sets and almost all t. The problem reduces to two integrodifferential equations which determine uniquely an additive function of sets $P(\tau, x; t, \Lambda)$ with $0 \leqq P \leqq 1$. In the case of an unbounded $p(t, x)$, however, $P(\tau, x; t, E)$ may fall short of unity, although all other requirements of the theory are always fulfilled. In the case of temporally homogeneous processes, necessary and sufficient conditions for $P(\tau, x; t, E)=1$ are given, which are related to the ergodic properties of the system. (Received January 28, 1940.)

170. Tomlinson Fort: *Summability and the definition of a limit.*

Let $S_0^x f(t)\Delta_\omega t$ denote a principal solution of the difference equation $\Delta_\omega y=f(x)$ as defined by Nörlund. In the present paper a study is made of the transformation $S_0^x K(x, t)f(t)\Delta_\omega t$ analogous to studies that have been made of the integral transformation $\int_0^\infty K(x, t)f(t)dt$. A variety of sufficient conditions are obtained that the transformation be regular, limit-producing, and so on. (Received January 18, 1940.)

171. H. L. Garabedian and H. S. Wall: *Hausdorff matrices and continued fractions.*

In a recent paper (abstract 46-1-130) Wall characterized totally monotone sequences in terms of continued fractions. In this paper the subclass of regular totally monotone sequences is so characterized. If $c_0, c_1, c_2, \cdots$ is totally monotone, then the row, column, and diagonal sequences of the difference matrix $(\Delta^m c_n)$ are totally monotone. Necessary and sufficient conditions are obtained in terms of the continued fraction in order that these be regular sequences. This is accomplished by means of the curious result that when $c_0-c_1x+c_2x^2-\cdots \sim c_0/1+g_1x/1+g_2(1-g_1)x/1+g_3(1-g_2)x/1+\cdots$, then $c_0-\Delta c_0x+\Delta^2c_0x^2-\Delta^3c_0x^3\cdots \sim c_0/1+(1-g_1)x/1+g_1g_2x/1+\cdots$, where the second continued fraction is obtained from the first by replacing g_{2n-1} by $1-g_{2n-1}$, $(n=1, 2, 3, \cdots)$. A hypergeometric summability obtained from the continued fraction of Gauss is investigated as to inclusion relations, and the methods of summability arising from the difference matrix for the base sequence are considered. Finally a general theorem is given on the effectiveness of the methods considered relative to analytic continuation of power series outside the circle of convergence. (Received January 13, 1940.)

172. G. A. Hedlund: *A new proof for a metrically transitive system.*

Two distinct methods have been used to prove that the flows defined by the geodesics on suitably restricted surfaces of constant negative curvature are metrically transitive. The first of these (cf. Annals of Mathematics, (2), vol. 35 (1934), p. 787) involves the use of symbolism to characterize the geodesics and is restricted to those surfaces for which a suitable symbolism has been devised. The second of these methods (cf. E. Hopf, *Ergodentheorie*) makes use of the theory of harmonic functions and is valid for all complete surfaces of constant negative curvature and of finite area. Both of the methods seem to involve excessive machinery and it would seem desirable to have a simpler and more straightforward proof of this result. The present paper gives a new and elementary proof of the metric transitivity of the flow defined by the geodesics on any closed orientable surface of constant negative curvature. The method can be readily extended to the case of complete surfaces of constant negative curvature and finite area. (Received January 25, 1940.)

173. G.A. Hedlund: *Surfaces of negative curvature and metric transitivity.*

Let U be the unit circle $z\bar{z}=1$, $z=x+iy$, and let Ψ be its interior. The author considers the quadratic form (A) $ds^2=\lambda^2(x, y)(dx^2+dy^2)/(1-x^2-y^2)^2$, $(x^2+y^2<1)$; where $\lambda(x, y)$ is a function defined in Ψ and satisfying the following conditions: (I) $\lambda(x, y)$ is of class C^7 in Ψ; (II) there exist positive constants a and b such that $a\leqq\lambda(x, y)\leqq b$ in Ψ; (III) there exist positive constants c and d such that if $K(x, y)$ denotes the Gaussian curvature of (A), $-d^2\leqq K(x, y)\leqq -c^2$ in Ψ; (IV) there exists a positive constant e such that if $H(P, Q)$ denotes the hyperbolic distance between arbitrary points P and Q of Ψ, then $|K(P)-K(Q)|\leqq eH(P, Q)$; (V) $\lambda(x, y)$ is invariant under a Fuchsian group which has U as principal circle, which is of the first kind, and which has a finite number of generators. If points which are congruent under F are considered identical, there is defined a manifold M of negative curvature. It is shown that the geodesic system on M is metrically transitive. The result has been attained previously only in the case when $K(x, y)$ is a negative constant. (Received January 25, 1940.)

174. Fritz Herzog: *Uniqueness theorems for rational functions.*

The uniqueness theorems, dealt with in this paper, are of the type of Nevanlinna's theorem which says that two meromorphic functions must be identical if they have *five* identical distributions, that is, if they assume five complex values (finite or infinite) at the same points. No assumption whatever is made about the multiplicity with which the two functions assume these five values at the various points. While five is the smallest number for which this uniqueness theorem holds true for meromorphic functions, the paper shows that for rational functions, *four* identical distributions are sufficient for the identity of the two functions. *Three* identical distributions, however, are not sufficient to insure the identity of the two functions. Several examples of pairs of rational functions with three identical distributions are given, also a triplet of such functions. Under certain restrictive conditions, concerning for instance the degrees of the two functions, uniqueness theorems for rational functions with three identical distributions are derived. It is also shown that two polynomials with identical distributions with respect to two finite values are identical. Finally, there can be no more than a finite number of rational functions with given distributions with respect to three given values. (Received January 6, 1940.)

175. F. B. Hildebrand and P. D. Crout: *A least square procedure for improving a method for solving integral equations by polynomial approximation.*

This paper continues the development of an approximate method for solving integral equations begun in a recent paper (Crout, Journal of Mathematics and Physics, Massachusetts Institute of Technology, vol. 19 (1940), pp. 34–92). This method has given very good results when applied to a number of physical problems, and consists essentially in replacing the unknown function by an algebraic polynomial determined by $2n+1$ unknown ordinates, using for convenience the Lagrangean form, tables for which are given. By interchanging the order of integration and summation and requiring that the integral equation be satisfied at a number of points, there is obtained a set of linear equations in the unknown ordinates which fix the polynomial. The least square process described is written into a single matrix multiplication, and gives a great increase in accuracy compared with the original technique; furthermore it permits a simple consideration of the error. In setting up physical problems and in applying the method described for solving the resulting integral equations, the work is greatly simplified by the continual appearance of matrices; also the work involved in any one problem automatically includes most of that required by a variety of other problems. (Received January 30, 1940.)

176. Einar Hille and I. E. Segal: *Blaschke products as Laplace-Stieltjes integrals.*

The possibility of representing a Blaschke product $B(z)$ of the half plane type as an absolutely convergent Laplace-Stieltjes integral in some half plane is investigated. It is shown that such a representation is impossible if the zeros of $B(z)$ are all real and tend to infinity, or if the zeros satisfy certain other conditions. A necessary and sufficient condition for $(z+1)^{-\alpha}B(z)$, $(0<\alpha\leqq 1/2)$, to be representable as an absolutely convergent Laplace-Stieltjes integral is given for the case where the zeros are all real and tend to infinity. The result is qualitatively that the more slowly the zeros tend to infinity, the larger α must be in order for $(z+1)^{-\alpha}B(z)$ to be representable as an absolutely convergent Laplace-Stieltjes integral in some half plane. (Received January 26, 1940.)

177. S. A. Jennings: *Nilpotent groups and nilpotent rings.*

Consider a nilpotent ring N which is free of operators and which satisfies an ascending chain condition for two sided ideals. Such a ring may be generated by a finite number of elements, and possesses a finite basis $n_1, \cdots, n_r$ so that any $n \varepsilon N$ may be written $n=\sum\alpha_i n_i$, where $\alpha_i n_i$ denotes $n_i+n_i+\cdots+n_i$ to α_i terms. Adjoin a unit element 1 to N and consider the set of elements of the form $1+n$. These elements form under multiplication a nilpotent group G, which is generated by a finite number of elements and whose class is not greater than the exponent of N. The order of G will be finite or infinite as the order of the additive group of N is finite or infinite. Conversely, let G be any nilpotent group with a finite number of generators. By using a result of W. Magnus (Journal für die reine und angewandte Mathematik, vol. 177 (1937), pp. 105–115) on the representation of the free group in a certain ring, it is shown that G determines a nilpotent ring of the above type, whose exponent is not greater than the class of G. There is, therefore, a 1:1 correspondence between nilpotent rings with a finite basis and nilpotent groups generated by a finite number of elements. (Received January 30, 1940.)

178. I. N. Kagno: *On a certain non-separating graph on an orientable surface.*

Let G be a connected linear graph, containing no vertices of degree less than 3, which is mapped on an orientable surface Σ_p of genus $p>0$ in such a manner that Σ_p-G is a single 2-cell. The structure of G is determined and it is shown that G has at most $6p-3$ arcs, and Σ_p-G has at most $12p-6$ sides. This result is applied to the universal covering surface of Σ_p and it is shown that a fundamental domain of the group of *decktransformations* of the covering surface has at most $12p-6$ sides. In particular, if Σ_p is an algebraic Riemann surface of genus $p>1$, the results of this paper can be applied to determine an upper bound for the number of sides of a fundamental domain of the group of linear transformations to which an automorphic function belongs. (Received January 19, 1940.)

179. E. R. Kolchin: *On the exponents of differential ideals.*

After defining the concept, "exponent of a differential ideal," theorems are given connecting the exponent of a differential ideal with the exponents of the factor ideals in the representation given by J. F. Ritt (Proceedings of the National Academy of Sciences, vol. 25 (1939), pp. 90–91) and showing the invariance of the exponent under differential field adjunction. These results are applied to the study of the exponent of the differential ideal generated by a single form in one unknown of order unity. It is seen that much depends on the nature of the singular solutions of the form. Complete results are not obtained, but the exponent is determined for a wide class of such ideals. Finally, there is a brief discussion of chains of differential ideals. A theorem is given relating the length to the exponent of a principal chain. (Received February 3, 1940.)

180. D. P. Ling and Leon Recht: *Geodesics on a paraboloid of revolution.* Preliminary report.

This paper is concerned with the study of the system of geodesics on a paraboloid of revolution. It is shown that the surface is divided into an infinite number of "zones" bounded by circles whose planes are perpendicular to the axis. The zone containing the vertex is characterized by the fact that no geodesic starting from a point in this zone can return to its starting point. Through every point in the next zone there is exactly one geodesic which returns to this point after one turn around the paraboloid, but none which returns after more than one turn. Through a point in the kth zone there are k geodesics which return after $1, 2, \cdots, k$ turns respectively, but none which returns after more than k turns. Finally, a method for approximating the positions of the circles between these zones is given. (Received January 19, 1940.)

181. J. K. L. MacDonald: *Existence theorems for node and amplitude properties near singularities of linear differential equations.*

Necessary and sufficient conditions for forms of solutions of linear, ordinary, second order differential equations near singular end points are established. Comparison, transformation and iteration theorems are generalized on the basis of previous work of the writer (this Bulletin, vol. 45 (1939), pp. 164–171, and abstract 44-9-356). Comparison criteria (analogous to the sequence of logarithmic tests for series convergence and divergence) are developed for determining whether or not nodes accumulate near the singular end points. Transformations leading in all cases to coefficients of the same sign are discussed. The phase angle and the transformed linear equations are expressed in forms which show their interrelationship in a clear way. (Received January 27, 1940.)

182. Saunders MacLane: *Note on the relative structure of p-adic fields.*

A problem of relative structure for p-adic fields, which arose in a previous paper (Annals of Mathematics, (2), vol. 40 (1939), pp. 423–442), can now be solved. Let k be a p-adic field with a residue class field $\mathfrak{k}$ of characteristic p. Let $\mathfrak{K}$ be any extension of this residue class field. A necessary and sufficient condition that any two p-adic extensions K and K' of k, both having the same residue class field $\mathfrak{K}$, be analytically equivalent over k is the requirement that the given residue class extension $\mathfrak{K}/\mathfrak{k}$ preserve p-independence (see definition in Duke Mathematical Journal, vol. 5 (1939), pp. 372–393). Here an *analytical equivalence* of K to K' is to be an isomorphism of K to K' which leaves values and residue classes of all elements in K fixed. The necessity proof is an extension of previous examples, and is based on a suitable analysis of those extensions $\mathfrak{K}/\mathfrak{k}$ which do not preserve p-independence. (Received January 29, 1940.)

183. R. S. Phillips: *On the space of completely additive set functions.*

This paper is devoted to an investigation of a class of Banach spaces typified by the space of completely additive set functions on a given Borel field of sets. The space of functions of bounded variation is a particular instance of such a space. The central result is that a space of this type can be represented as the direct product of spaces of completely additive set functions, each absolutely continuous with respect to a unique measure function of a certain orthogonal set. The author shows that to each set function $x(\tau)$ of the space corresponds a denumerable number of disjoint sets τ_i such that $x(\tau)$ is uniquely representable in the form $x(\tau)=\sum x(\tau\cdot\tau_i)$ where $x(\tau\cdot\tau_i)$ belongs to one of the component spaces. By means of this decomposition it is easy to demonstrate that the space is weakly complete and that it possesses a generalized base. Finally the author is able to define the general linear functional on the space and characterize the conjugate space. (Received January 9, 1940.)

184. Tibor Radó and P. V. Reichelderfer: *Note on an inequality of Steiner.*

Let $L(f)$ denote the Lebesgue area of a continuous surface $z=f(x, y)$ defined over the unit square Q. If $z=f_n(x, y)$ is a sequence of continuous surfaces defined over Q and converging uniformly to $z=f(x, y)$ on Q for which $L(f_n)$ and $L(f)$ are finite, then McShane (Annals of Mathematics, (2), vol. 33 (1932), pp. 125–138) has shown that smallness of $\epsilon_n\equiv[L(f_n)+L(f)]/2-L([f_n+f]/2)$ implies smallness with respect to measure of $\delta_n(x, y)\equiv[(f_{nx}-f_x)^2+(f_{ny}-f_y)^2]^{1/2}$, whence he concludes that if $L(f_n)$ converges to $L(f)$, then $\delta_n(x, y)$ converges to zero in measure. This result is improved in the present paper by replacing smallness with respect to measure by smallness with respect to exponent. It is shown that smallness of ϵ_n implies smallness of $I_n(\lambda)\equiv\iint_Q\delta_n^\lambda(x, y)dxdy$ for every exponent λ satisfying $0<\lambda<1$, and thus if $L(f_n)$ converges to $L(f)$ then $I_n(\lambda)$ converges to zero for every λ satisfying $0<\lambda<1$. Since McShane has shown that $I_n(1)$ does not generally converge to zero when $L(f_n)$ converges to $L(f)$, the present result is the best obtainable under the stated hypotheses. (Received January 19, 1940.)

185. Eric Reissner: *On Green's function of* $\nabla^2\nabla^2 w=0$ *for the half plane.*

The following result, of interest mainly for the theory of bending of thin elastic plates, is proved. The explicit form of Green's function of $\nabla^2\nabla^2 w=0$ for the half plane $x \geqq 0$ satisfying the boundary conditions $w=0$ and $(\partial w/\partial x)=\lambda\nabla^2 w$ along $x=0$ is: $w=(1/8\pi)\Re n\left[|z-a|^2\log(z-a)/(z+a)+2az+8ax\exp((z+a)/2\lambda)\,\mathrm{Ei}((z+a)/2\lambda)\right]$. In the formula for w, Ei (x) denotes the "exponential integral." The solution for the special case $\lambda=0$ has been given by J. H. Michell (Proceedings of the London Mathematical Society, vol. 34 (1902), pp. 223–228). (Received January 25, 1940.)

186. H. J. Riblet: *Certain theorems for algebraic differential fields.*

In this paper several theorems are proved for finite algebraic differential fields. One states that in a finite algebraic differential field $\overline{K}(\theta_1)$ there exists an element which is linearly independent of its $n-1$ conjugates. This is helpful in proving that if every element of $\overline{K}(\rho)$ satisfies a linear homogeneous differential equation of order not greater than n over $\overline{K}$, then ρ is algebraic over $\overline{K}$ of degree not greater than n. (Received January 16, 1940.)

187. Jenny E. Rosenthal: *Generating functions and properties of certain orthogonal polynomials.* Preliminary report.

The differential equation satisfied by the generating functions $U(x, t) =\sum_{m=0}^{\infty}R_m(x)t^{m+\alpha}$ of a set of orthogonal polynomials is found to be $\sum_{j=0}^{n}t^j\left[a_j+(b_jx+c_j)t+d_jt^2\right]\cdot\partial^jU/\partial t^j=0$, where the various a's, b's, c's, and d's are constants. Depending on the values of these constants, the solution of the equation for $n-1$ gives three new functions in addition to the generating functions of the Legendre, Laguerre, and Hermite polynomials. The properties of the orthogonal polynomials generated by the new functions are investigated. The range of orthogonality and the weight factors are derived by a simple method in two of the cases. The weight factors are of the form of gamma- and beta-functions. Possible generalizations of the results are discussed. (Received January 27, 1940.)

188. Peter Scherk: *Estimations with integers in the demesne of the α-β-hypothesis.*

The "number function" $D(x)$ of a set D of positive integers d denotes the number of the d not greater than x. Let A, B be sets of positive integers respectively a, b; and let C be the set of all the numbers of the form a, b or $a+b$. The author gives two new estimates of the number function of the set C through those of A and B. One of them is the simplest possible symmetric in A and B, the other being related to it by Khintchine's "Umkehrformel." Both are exact. The proofs are based on a classical special case of this formula. The methods are similar to those used in two earlier papers by Besicovitch and by the author. See Landau's Cambridge Tract and the author's *Bemerkungen zu einer Note von Besicovitch*, Journal of the London Mathematical Society, vol. 14 (1939), pp. 185–192. (Received January 19, 1940.)

189. F. K. Schmidt and Saunders MacLane: *On inseparable fields.*

The original proof of the structure theorem for p-adic fields depended on an analysis of the residue class field by means of certain Steinitz field towers. Saunders Mac Lane has given examples to show that the original analysis of these towers could not be correct (Transactions of this Society, vol. 46 (1939), pp. 23–45). The authors now develop an alternative treatment of modified Steinitz towers, in terms of which the proof of the structure theorem can be carried out. The first modification is the device of replacing a denumerable transcendence basis $t_0, t_1, t_2, \cdots$ by another basis $t_0, t_1^p, t_2^{p^2}, \cdots$, which is almost a p^∞th power. The second modification consists in generating an arbitrary extension K preserving p-independence over a modular field

k by a transfinite sequence of extensions, each of which has a denumerable transcendence basis and preserves p-independence. (Received January 29, 1940.)

190. A. R. Schweitzer: *Concerning general abstract relational spaces.*

The author constructs a set S of postulates for relational spaces in terms of an undefined transitive and symmetrical relation K_H effective between $(n+1)$-ads of elements $(n=1, 2, 3, \cdots)$ and invariant under any arbitrarily selected subgroup H of the symmetric group G on $n+1$ elements, including G. When H is the alternating group and $n=3$ the set S is equivalent to Axioms 1–6, 11–13, 16 of the author's descriptive system ${}^3K_3$ (American Journal of Mathematics, vol. 31 (1909), pp. 395–396). A feature of the general theory is the generalized axiom of dimensionality (generalizing Axiom 16, above) which expresses disjunctively relative classification of $(n+1)$-ads corresponding to the classification of the substitutions of the symmetric group G on $n+1$ elements into the subgroup H and cosets with respect to H. Particular instances of the preceding relational spaces are rotation spaces of the regular polygons and polyhedra and other metric spaces. When H coincides with G, instances include the author's system of postulates ${}^nK^*$ for projective geometry (this Bulletin, abstract 42-11-443). (Received January 28, 1940.)

191. W. T. Scott: *Approximation to real irrationals by certain classes of rational fractions.*

The set of all irreducible rational fractions is divided into three classes, according as the numerator and denominator are even or odd integers. For each of these classes it is found that there are infinitely many fractions p/q satisfying $|\omega-p/q|<k/q^2$ when $k\geqq 1$, regardless of the value of the real irrational number ω. If $k<1$, there exist irrationals everywhere dense on the real axis for which the inequality holds for at most a finite number of fractions of a given one of the three classes. The method of proof depends on the geometric properties of elliptic modular transformations. (Received January 6, 1940.)

192. Seymour Sherman: *A comparison of linear measures of point sets in the plane.* Preliminary report.

The purpose of this investigation is to compare the linear measures defined by Carathéodory and Deltheil. H. Steinhaus (Comptes Rendus du Premier Congrès des Mathématiciens des Pays Slaves, 1929, pp. 348–354) has proved that the Deltheil measure of a rectifiable plane Jordan curve is twice its length and so twice its Carathéodory linear measure. Here it is shown that, contrary to Steinhaus's belief, there exist sets (in particular every set irregular in the sense of Besicovitch) with Deltheil measure zero and Carathéodory measure nonzero. In general if a set has finite Carathéodory linear measure, then it is Deltheil measurable with Deltheil measure equal to twice the Carathéodory linear measure of its regular part. Deltheil measurability does not imply linear measurability in the sense of Carathéodory. Nothing is proved concerning sets of noncountably infinite Carathéodory linear measure and no extensions are made to sets in more general product spaces or even to sets in higher dimensional euclidean spaces. (Received January 11, 1940.)

193. M. F. Smiley: *Measurability and modularity in the theory of lattices.*

In a previous note (*A note on measure functions in a lattice*, to appear in this Bulletin), a notion of measurability (with respect to a function μ) of elements of an arbitrary lattice was introduced. Results of V. Glivenko (*Contributions à l'étude des systèmes de choses normées*, American Journal of Mathematics, vol. 59 (1937), pp. 933–934) and of L. R. Wilcox and the author (*Metric lattices*, Annals of Mathematics, (2), vol. 40 (1939), p. 313) indicate that the idea of measurability and that of modularity are intimately related. The purpose of this note is to exhibit a further relationship which does not depend on metric properties of the function μ. Under the assumption that $a \leqq b$ with $\mu(a) = \mu(b)$ implies that $a = b$, it is found that the measurability of each subelement of a fixed element c implies the complete modularity of each subelement of c. A further consequence is the proof of a theorem in the reverse direction in case the lattice is of finite dimension and satisfies the conclusion of Jordan's theorem. (Received February 1, 1940.)

194. Gabor Szegö: *Remarks on a note of R. Wilson and related subjects.*

Let $w(x)$ be a weight function in $[-1, +1]$ and $\{p_n(x) = k_n x^n + \cdots\}$ the associated set of orthonormal polynomials. (1) Assuming the existence of $\int_0^\pi \log w(\cos\theta)d\theta$, a simple proof is given for the relations $\lim_{n\to\infty} \max |p_n(x)|^{1/n} = (1/2)\cdot\lim_{n\to\infty} k_n^{1/n} = 1$, $-1 \leqq x \leqq +1$, recently discussed by R. Wilson (this Bulletin, vol. 45 (1939), p. 190). (2) An older theorem of Shohat stating that the equivalency of the relations $2^{-n}k_n = O(1)$, $\lim_{n\to\infty} 2^{-n}k_n$ exists is proved in an arrangement slightly different from that of Shohat. (3) It is shown that the relation $2^{-n}k_n = O(1)$ (or the existence of $\lim_{n\to\infty} 2^{-n}k_n$) is equivalent to the existence of the integral mentioned in (1). (Received January 22, 1940.)

195. G. B. Thomas: *Regular positive ternary quadratic forms.* Preliminary report.

Let $f = f(x, y, z) = ax^2 + by^2 + cz^2 + 2ryz + 2sxz + 2txy$ be a positive, reduced, properly or improperly primitive form with integral coefficients and Hessian $H = \Omega^2\Delta$, where Ω is the g.c.d. of the 2-rowed minors of H. Then f is said to be *regular* if the integers not represented by f coincide with the positive integers in certain sets of arithmetic progressions. (L. E. Dickson, Annals of Mathematics, (2), vol. 28 (1927), pp. 333–341.) The problem of determining all such regular forms with $r = s = t = 0$ has been solved. (Dickson, loc. cit.; B. W. Jones and Gordon Pall, Acta Mathematica, vol. 70 (1939), pp. 165–191.) Combining some results due to B. W. Jones (Transactions of this Society, vol. 33 (1931), pp. 92–124) with certain theorems regarding primes in arithmetic progressions, the present paper shows that there are not more than 99 nor fewer than 94 regular forms with $\Omega = 1$ and r, s, t not all zero. The regular forms are listed, together with the progressions of integers they do not represent. (Received January 16, 1940.)

196. Leonard Tornheim: *Linear forms in function fields.*

The following analogue for function fields of a well known theorem of Minkowski on linear forms is proved algebraically. Let L_i be n linear expressions in n variables x_j with coefficients a_{ij} in $F(z)$ where z is an indeterminate over a field F and with determinant $|a_{ij}|$ of degree d. If the sum of n integers c_i is greater than $d - n$, there exists a set of values for the x_j in $F[z]$ such that each L_i has degree at most c_i. (Received January 15, 1940.)

197. H. S. Vandiver: *Elements of a theory of abstract discrete semi-groups.*

A set of axioms is given for a semi-group, the latter being a system elsewhere roughly described by the author as a set closed under an associative operation. A feature of the treatment employed is the fact that the parentheses are not used, there being an axiom of substitution which leads to a result which is equivalent to the associated law, but which does not contain parentheses in its statement. The symbol of relation ($\stackrel{\frown}{=}$) (called equivalence) does not necessarily stand for equality in the usual sense; for example, the integers 1, 2, 3, 4 are said to form a semi-group under multiplication, modulo 5. In this case equivalence means "is congruent to, modulo 5." The proofs of a number of theorems given without proof in a former article (Proceedings of the National Academy of Sciences, vol. 23 (1937), pp. 552–555) are included in this paper. A theorem is also proved concerning semi-groups, which includes Dedekind's well known theorem of inversion. (Received December 30, 1939.)

198. H. S. Vandiver: *On general methods for obtaining congruences involving Bernoulli numbers.*

This paper has been published (this Bulletin, vol. 46 (1940), pp. 121–123).

199. Abraham Wald: *Asymptotically most powerful tests of statistical hypotheses.*

Let $F(x, \theta)$ be the distribution of a variate X involving an unknown parameter θ. For testing the hypothesis $\theta=\theta_0$, it is necessary to choose a region of rejection W_n in the n-dimensional sample space. Denote by $P(W_n|\theta)$ the probability that the sample point will fall in W_n under the assumption that θ is the true value of the parameter. For any region U_n denote by $g(U_n)$ the greatest lower bound of $P(U_n|\theta)$. For any pair of regions U_n and T_n denote by $L(U_n, T_n)$ the least upper bound of $P(U_n|\theta)-P(T_n|\theta)$. An infinite sequence $\{W_n\}$ of regions is said to be an asymptotically most powerful test of the hypothesis $\theta=\theta_0$ on the level of significance α if $P(W_n|\theta_0)=\alpha$ and if for any sequence $\{Z_n\}$ of regions for which $P(Z_n|\theta_0)=\alpha$, lim sup $L(Z_n, W_n)\leqq 0$ holds. A sequence of regions $\{W_n\}$ is said to be an asymptotically most powerful unbiased test of the hypothesis $\theta=\theta_0$ on the level of significance α if $P(W_n|\theta_0)=\alpha$, lim inf $g(W_n)\geqq\alpha$, and if for any sequence $\{Z_n\}$ of regions for which $P(Z_n|\theta_0)=\alpha$ and lim inf $g(Z_n)\geqq\alpha$ the inequality lim sup $L(Z_n, W_n)\leqq 0$ holds. It has been shown that the commonly used tests, based on the maximum likelihood estimate, are asymptotically most powerful. (Received February 5, 1940.)

200. William Wernick: *Complete sets of logical functions.*

H. M. Sheffer has shown that the 16 functions of the calculus of sentences can all be defined by the single function "stroke" which may have a dual interpretation. Other authors have shown that certain pairs of undefined functions can define this calculus, and that certain other pairs cannot. The set of all functions which can be defined in terms of members of a given undefined set of functions is called the field of that given set. Fields of all single functions and of pairs, triads, and tetrads of functions are discussed, using known properties of symmetry, duality, and transitivity. A set of undefined functions is called "*complete*" if it is *sufficient* (all the 16 functions can be defined in terms of members of the set) and *non-redundant* (no member of the set can be defined in terms of other members of the set). The existence of 2 complete sets of single functions, 24 complete pairs, and 6 complete triads is shown. It is proved that there exists no complete set of four or more functions, by showing that if such a set is sufficient it must be redundant. (Received January 24, 1940.)

201. Hermann Weyl: *Theory of reduction for arithmetical equivalence.*

In the frame of Minkowski's geometry of numbers, reduction appears as the problem to normalize a basis for the n-dimensional lattice of vectors with integral components in terms of a given convex body. The theory in this generality can be carried through much further than Minkowski seems to have assumed, by means of his geometric methods. He limited himself to the special case of ellipsoids (quadratic forms), at the same time abandoning his own geometric approach. Even for this special case the new treatment induces considerable simplification, closes certain gaps, and results in sharper estimates. The theory is carried over to vectors whose components are complex numbers or quaternions, with an integrity domain of class number 1 serving for the definition of lattice vectors. The quadratic forms are then replaced by Hermitian and "Hamiltonian" forms respectively. (Received January 24, 1940.)

202. John Williamson: *An algebraic problem involving the involutory integrals of linear dynamical systems.*

It is known that a conservative Hamiltonian system with n degrees of freedom has exactly n independent conservative integrals which form an involutory system. It is here shown that, if the system is linear, these n integrals may be chosen as quadratic forms. It is also shown that, if $2H$ is the symmetric matrix, which is the Hessian of the Hamiltonian function, and G the matrix of the covariant bilinear form, that is, the standard canonical form of a nonsingular skew symmetric matrix, the total number of independent conservative quadratic integrals is $2n-m$, where either $2m$ or $2m+1$ is the order of the minimal equation of HG^{-1}. In case HG^{-1} is nonderogatory, the n independent quadratic integrals may be so chosen that their matrices are the symmetric matrices $(HG^{-1})^{2k}H$, $(k=0, 1, 2, \cdots, n-1)$. These n integrals in this case form a set in involution. (Received January 30, 1940.)