THE BASES OF PROBABILITY[1]

B. O. KOOPMAN

The subject for consideration today forms an aspect of a somewhat venerable branch of mathematical theory; but in essence it is part of a far older department of thought—the ancient science of logic. For it is concerned with a category of propositions of a nature marked by features neither physical nor mathematical, but by their rôle under the aspect of the reason. Their essential characteristic is their involvement of that species of relation between the knower and the known evoked by such terms as probability, likelihood, degree of certainty, as used in the parlance of intuitive thought. It is our threefold task to transcribe this concept into symbols, to formulate its principles, and to study its properties in their inner order and outward application.

As prelude to this undertaking it is necessary to set forth certain conventions of logic. Propositions are the elements of symbolic logic, but they may play the rôle of *contemplated propositions* (statements in quotation marks) or of *asserted propositions* (statements regarded as true throughout a given manipulation or deduction); and it is necessary to take account of this in the notation for the logical constants. We shall employ the symbols for negation ($\sim$), conjunction or logical product ($\cdot$), and disjunction or logical sum ($\vee$), and regard them as having no assertive power: they combine contemplated propositions into contemplated propositions and asserted propositions into asserted propositions *of the same logical type*. Quite other shall be our convention regarding implication ($\subset$) and equivalence ($=$): they combine contemplated propositions into asserted propositions, and shall not be used to combine asserted propositions in our present study. If a and b stand for contemplated propositions, the assertion that a is false (true) shall be written $a=0$ ($a=1$), and the assertion that a implies b, $a \subset b$ or $a\sim b=0$; it is thus quite different from the contemplated proposition $\sim(a\sim b)$. Finally it is universally asserted that $a\sim a=0$, $a \vee \sim a=1$; and in fact all the laws of Boolean algebra are regarded as assertions. We shall assume their elements to be fa-

[1] An address delivered before the New York meeting of the Society on February 24, 1940, by invitation of the Program Committee.

For the details of the theory here expounded, see the two publications of the present author. *The axioms and algebra of intuitive probability*, Annals of Mathematics, (2), vol. 41 (1940), pp. 269–292 (herein to be abbreviated as AAP) and *Intuitive probability and sequences* (forthcoming in the Annals of Mathematics) (abbreviation PS).

miliar, and shall accept without question the intuitive logical background implied in their manipulation and interpretation.[2]

The technical logician will observe that there is here involved a meta-mathematical question. But space forbids us to introduce and explain here the modern terminology of this subject, and compels us to throw a perhaps undue burden on the terms "contemplated" and "asserted proposition."

The next step towards our goal consists in defining a category of propositions which are to form the substratum upon which the structure of our theory is to be erected. We will designate by *experimental propositions* such statements of the outcome of a particular physical or biological event as may in principle be verified by the performance of a single crucial experiment. Thus "it will rain on this roof tomorrow at this hour" or "Mr. X made a mistake in his accounts last Monday" are experimental propositions, whereas "Newtonian mechanics is correct" is not: the motion of bodies can always be accounted for by assuming sufficiently complicated laws of force in Newton's equations, so its truth, while experimental in meaning, is not determined by a crucial experiment but rather by its ability to harmonize the results of many such experiments in an acceptably simple manner. Experimental propositions shall be denoted by lower case Latin letters; and inasmuch as a finite set of crucial experiments may always be regarded as constituting a single crucial experiment, finite combinations of letters by means of $(\sim \cdot \vee)$ also denote experimental propositions. Finally, experimental propositions shall be regarded as contemplated propositions.[3]

At this point the category of propositions may be introduced which forms the subject of the theory of probability. We will take as the conceptual germ from which the whole theory springs the ordering of two events in the relation "not more probable than," a relation to be denoted by the partial ordering symbol $(\prec)$—which like $(\subset, =)$ shall have assertive force. One could of course develop a theory of such assertions as $a \prec b$; but it would prove insufficient for the purposes of probability, as one needs for example to compare the probability of a assuming h true with its probability assuming h false. The definitive form of propositions sought is the following, in which a, b, h, k are experimental propositions and $h \neq 0$, $k \neq 0$:

[2] For references see E. V. Huntington, Transactions of this Society, vol. 35 (1933), pp. 274–304.

[3] In its occurrence in a/h the experimental proposition h is in a sense "temporarily asserted," i.e., a is viewed on the assumption that h is true. But we are applying the term asserted proposition only to those held as true on *both* sides of $a/h \prec b/k$.

a on the presumption h is no more
probable than b on the presumption k

and this is symbolized as $a/h \prec b/k$ (or equivalently, $b/k \succ a/h$). Thus from the four contemplated experimental propositions and the logical symbol $(/ \prec /)$ an asserted proposition $a/h \prec b/k$ (to be called a $(\prec)$ proposition) is formed which is to constitute the building stone of the whole theory of probability.

Before proceeding further, two difficulties must be surmounted. Firstly it may be objected that $a/h \prec b/k$ is a proposition of such vague and subjective order that it may not merely be held by one person and rejected by another, but that one and the same individual may sometimes assent to it and at other times and in a different mood reject it. If this is so, how in the nature of things can such propositions form the subject matter of a precise mathematical science? Secondly it may be objected that the probability of a proposition a depends on a body of knowledge going far beyond the fact that h is true: It will involve propositions of higher logical types such as the laws of logic—and of probability itself; and perhaps even matters of subconscious moods, associations, artistic taste, and the like.

We believe that these two objections are answered at one stroke by adhering to the following convention, or rather, clarification of the use and laws of $(\prec)$ propositions. *A given individual at a given moment* may be regarded as assenting to a certain set of $(\prec)$ propositions; ignoring what he may hold at any other moment or what others may believe, that set of $(\prec)$ propositions which he holds at that given moment must have certain relations with one another which may be called *relations of consistency*. It is to their formulation and study that we conceive the present science to be devoted. So viewed, the analogue with strict logic is clear: many may disagree with me when I assert $a=0$; but every one whose mind is constructed on normal lines will agree that if $a=0$ then $(\sim\sim a)=0$. Along with the first objection, this convention answers the second, for by positing a given individual at a given moment in the consideration of any set of $(\prec)$ propositions, the body of knowledge becomes fixed throughout, and so does not require explicit symbolization.

A third objection which might be voiced is against the restriction of the application of probability to experimental propositions. Why can we not compare the probability of two physical theories, for example? In answer to this we can say only that with the present restriction many grave logical difficulties are avoided and a theory is obtained which covers all the classical cases of mathematical probability and many others as well; and our limitation of the scope is a matter of practical strategy rather than of principle. It would undoubtedly be of interest to extend the present ideas to propositions of higher logical types, classes of propositions, logical systems and the like.

There is evidently no difference in intuitive meaning between a/h and ah/h (or, dually, $a \vee \sim h/h$), and we will make the notational convention that any one of these symbols be replaceable by any other. This has an interesting algebraic counterpart. Let $\mathfrak{A}$ be the Boolean ring determined by all the experimental propositions considered in a given discussion.[4] To make the presumption that $h \varepsilon \mathfrak{A}$ is true (equivalently, that $\sim h$ is false) is to make the presumption that any two propositions a and b of $\mathfrak{A}$ for which $a\sim b \subset \sim h$ and $b \sim a \subset \sim h$ are equivalent, i.e., both true or false simultaneously. But this "identification" of all so-related pairs a, b is precisely the formation of the quotient ring $\mathfrak{A}/(\sim h)$ whose elements $a/(\sim h)$ are the remainder classes with respect to the principle ideal $(\sim h)$. It is purely for convenience that we write a/h in lieu of $a/(\sim h)$. Thus if $\mathcal{A}$ is the class of all remainder classes in $\mathfrak{A}$ with respect to all its principal ideals, the $(\prec)$ symbol introduces a partial ordering of the elements of $\mathcal{A}$.[5]

We are now ready to undertake our second task and lay down the axioms which govern any aggregate of $(\prec)$ propositions. It will be noted that they all have $(\subset, =)$ or $(\prec)$ propositions as hypothesis and as conclusion, and that in each case where the conclusion is a non-trivial $(\prec)$ proposition, this is true of the hypothesis as well. Finally, a tacit assumption is always made: *no denominator* $=0$.

THE AXIOMS

V. AXIOM OF VERIFICATION. *If* $k \subset b$, *then* $a/h \prec b/k$.

I. AXIOM OF IMPLICATION. *If* $a/h \prec b/k$ *and* $h \subset a$, *then* $k \subset b$.

R. AXIOM OF REFLEXIVITY. *If* $h=k$ *and* $ah=bk$, *then* $a/h \prec b/k$.

T. AXIOM OF TRANSITIVITY. *If* $a/h \prec b/k$ *and* $b/k \prec c/l$, *then* $a/h \prec c/l$.

A. AXIOM OF ANTISYMMETRY. *If* $a/h \prec b/k$, *then* $\sim a/h \succ \sim b/k$.

C. AXIOMS OF COMPOSITION. *Let* $0 \neq a_1 \subset b_1 \subset c_1$ *and* $0 \neq a_2 \subset b_2 \subset c_2$.

[4] For Boolean rings and their ideals, see M. H. Stone, Transactions of this Society, vol. 40 (1936), pp. 37–111.

[5] One should guard against the notion that this ordering has any simple relation with the ordering of the remainder classes with respect to class inclusion.

C_1. *If* $a_1/b_1 \prec a_2/b_2$ *and* $b_1/c_1 \prec b_2/c_2$, *then* $a_1/c_1 \prec a_2/c_2$.
C_2. *If* $a_1/b_1 \prec b_2/c_2$ *and* $b_1/c_1 \prec a_2/b_2$, *then* $a_1/c_1 \prec a_2/c_2$.

D. AXIOMS OF DECOMPOSITION (QUASI-CONVERSES OF C). *Let* $0 \neq a_1 \subset b_1 \subset c_1$, $0 \neq a_2 \subset b_2 \subset c_2$, *and* $a_1/c_1 \prec a_2/c_2$. *Then if either symbol in* (i): $(a_1/b_1, b_1/c_1)$ *has the* $(\succ)$ *relation with either in* (ii): $(a_2/b_2, b_2/c_2)$, *then the remaining symbol in* (i) *has the* $(\prec)$ *relation with that in* (ii). (*Thus* D *contains four axioms.*)

P. AXIOM OF ALTERNATIVE PRESUMPTION. *Let* $a/hb \prec r/s$ *and* $a/h{\sim}b \prec r/s$; *then* $a/h \prec r/s$.

S. AXIOM OF SUBDIVISION. *For each positive integer n the following axiom is posited*:
S_n. *If* $a_1 \vee \cdots \vee a_n = a \neq 0$, $b_1 \vee \cdots \vee b_n = b \neq 0$, $a_i a_j = b_i b_j = 0$ *for all* $i \neq j$, *and lastly if*

$$a_1/a \prec a_2/a \prec \cdots \prec a_n/a,$$
$$b_1/b \succ b_2/b \succ \cdots \succ b_n/b,$$

then $a_1/a \prec b_1/b$.

The Axioms V, R, T, A are simply the transcription into the present language of facts so familiar as scarcely to require comment. The partial ordering property of $(\prec)$ expresses itself by R and T; it leads to the definition of equiprobability $a/h \approx b/k$, inferior probability $a/h < b/k$ and incomparability $a/h \| b/k$ in the usual manner. The question of whether one can go further and assume that the entities a/h form the elements of a lattice with respect to $(\prec)$ will naturally be raised; until now we have been unable to make any use of this idea, and if our experience is borne out we shall be in the presence of the first non-trivial example of a partially ordered set which is not a lattice.

Axiom I is in sharp contrast with the familiar circumstance that the numerical probability of an event may be unity (i.e., the same as a certain event) without that event's being certain. This is because numerical probability gives but a blurred rendering of the ultimate logical relations between probability and certainty.

As for Axiom C and its converse D, the following verbal rendering of C_1 may be given: If a_1 depends for its possibility on b_1, and likewise a_2 on b_2, and if c_1 is less likely to lead to b_1 than is c_2 to b_2, and if finally b_1 is in turn less likely to lead to a_1 than is b_2 to a_2, then c_1 is less likely to lead to a_1 than c_2 to a_2. So stated, it exhibits a sort of inner transitivity. All the other cases have a corresponding phraseology.

One might be disposed to regard Axiom P as a theorem which could be proved by arguing that since the hypothesis $a/hb \prec r/s$, $a/h{\sim}b \prec r/s$ tells us that a on the presumption h is not more likely than r/s both when b is true and when b is false, it must be so in all cases, i.e., $a/h \prec r/s$. Carrying this species of reasoning a little further, we could prove that such a relation as $a/h \approx {\sim}a/h$ is impossible: Since the assertion $(a \vee {\sim}a) = 1$ is always made we would conclude that the only possibilities are $a = 1$ or ${\sim}a = 1$ (i.e., $a = 0$); in either case $a/h \approx b/k$ is impossible—hence it is never possible. In essence this is the old objection of the elementary student who voices it by saying that since an event will either happen or not happen, it is absurd to say that its probability of happening could ever be $\frac{1}{2}$. We all know how to answer him by general reference to the dependence of probability on a body of knowledge;[6] but we are in a position here to give the answer in a precise logical form: The fallacy lies in confusing the assertion $(a \vee {\sim}a) = 1$ with the assertion "$a = 1$ *or* ${\sim}a = 1$" [which might be written $(a = 1) \vee ({\sim}a = 1)$]. *The distinction between an asserted disjunction and a disjoined assertion is fundamental*: $(u \vee v) = 1$ *must never be confused with* $(u = 1) \vee (v = 1)$. The disregard of this distinction has led to more difficulties in the foundations of probability than is often imagined. It is now clear that the above proof of Axiom P is fallacious since it confuses $(b \vee {\sim}b) = 1$ with $(b = 1) \vee ({\sim}b = 1)$. As a matter of fact the same proof would have provided an infinite extension of Axiom P, an extension which leads to paradoxes.[7]

Axiom S is epitomized in the idea that if a first event is less likely of occurrence than its opposite and if a second is more likely than its opposite, then the first is less likely than the second. While we have not succeeded in simplifying the general case (beyond restricting S_n to prime values of n), we still feel confident that those more skilful than ourselves may have better success.

As a purely formal matter it may be remarked that in a system *completely* ordered by $(\prec)$, Axioms P and S are logical consequences of the rest.

Our second task being complete, we pass to the third, the deduction of all the useful theorems of probability from the axioms. But before proceeding it may be remarked that we have traversed the path of all mathematical disciplines: One proposes to study a subject of which

[6] The notion that the uncertainty resides in the events themselves rather than in the mind of the individual contemplating them, the appeal to the "principle of uncertainty," etc., betrays merely a misconception both of probability and of quantum mechanics.

[7] Cf. PS, §2.

one is made aware through the intuition, the senses, and such non-mathematical modes of perception. In undertaking it one introduces symbols and statements of axioms and laws in terms of these. But from this point on everything (except the interpretation) becomes purely mathematical: The symbolic abstractions become the ultimate objects of study, the axioms and laws become the postulates. Thenceforth we may discard the intuitionalistic introduction of our symbols and axioms and regard the latter as pure conventions (postulates) pertaining to the former, which are taken as "undefinables."

The further developments fall into three groups of theorems: the theorems on comparison, the theorems on numerical probability, and the theorems on statistical weight or frequency in a sequence.

In the first group we shall confine ourselves to citing the following typical ones. No comments appear necessary.

THEOREM. *If* $ah \neq 0$ *or* h, *then* $0/1 < a/h < 1/1$.

THEOREM. *If* $a_1/h_1 \prec a_2/h_2$, $b_1/h_1 \prec b_2/h_2$, *and* $a_1b_1h_1 = a_2b_2h_2 = 0$, *then* $a_1 \vee b_1/h_1 \prec a_2 \vee b_2/h_2$.

THEOREM. *If* $a/hc_i \prec r/s$ $(i = 1, \cdots, n)$ *and* $hc_ic_j = 0$ *for all* $i \neq j$, *then* $a/hc \prec r/s$ *where* $c = c_1 \vee \cdots \vee c_n$.

The second group starts with the introduction of the numerical probability $p(a/h) = p(a, h)$, i.e., the number between 0 and 1 forming the basis of the classical theory. This is accomplished as follows:

DEFINITION. *Any set of propositions* $(u_1, \cdots, u_n)$ *shall be called an* n*-scale when they satisfy the conditions* (i) $u_1 \vee \cdots \vee u_n = u \neq 0$; (ii) $u_iu_j = 0$ *(all* $i \neq j$*)*; (iii) $u_i/u \approx u_j/u$ *(all* i, j*)*.

ASSUMPTION. *Any positive integer* n *being given, the conceptional existence of at least one* n*-scale may be assumed.*

This is the only principle which need be assumed in addition to the axioms in all the further developments of the theory.[8] It is of a fundamentally different nature from the axioms and might be compared with the assumption so familiar in thermodynamics and other parts of physics of the possibility of a conceptual experiment.

THEOREM. *If* $(u_1, \cdots, u_n)$ *is an* n*-scale and* $(v_1, \cdots, v_m)$ *an* m*-scale*,

$$u_1 \vee \cdots \vee u_\nu/u \ <, \approx, > \ v_1 \vee \cdots \vee v_\mu/v$$

according as $\nu/n \ <, =, > \ \mu/m$.

[8] In the exhibition of certain paradoxes the extension of the assumption regarding the existence of n-scales to $n = \aleph_0$ is required.

Let $t(n)$ be the maximum value of t for which $u_1 \vee \cdots \vee u_t/u \prec a/b$ holds ($t=0$ corresponding to $0/u \prec a/b$) and $T(n)$ the minimum value of T for which $a/b \prec u_1 \vee \cdots \vee u_T/u$. By the previous theorem $t(n)$ and $T(n)$ are independent of the particular n-scale chosen; for fixed a/b they are always defined functions of n. We then prove the following:

THEOREM. *The limits $p_*(a/h) = \lim_{n\to\infty} t(n)/n$ and $p^*(a/h) = \lim_{n\to\infty} T(n)/n$ always exist, and $0 \leqq p_*(a/h) \leqq p^*(a/h) \leqq 1$.*

The numbers $p_*(a/h)$ and $p^*(a/h)$ may be called the lower and upper numerical probabilities of a/h.

DEFINITION. *If $p_*(a/h) = p^*(a/h)$, a/h is said to be appraisable and to have $p(a/h) = p_*(a/h) = p^*(a/h)$ as its numerical probability.*

All the classical theorems follow. We give merely the following two examples:

THEOREM. *Let $a \subset b \subset c$; if a/c and b/c are appraisable and if $p(b/c) \neq 0$, then a/b will be appraisable and $p(a/c) = p(a/b)p(b/c)$.*

THEOREM. *Let a/h and b/h be appraisable. Then $a \vee b/h$ will be appraisable if and only if ab/h is appraisable and it will then follow that*

$$p(a/h) + p(b/h) = p(a \vee b/h) + p(ab/h).$$

If now we consider the Boolean ring determined by the totality of propositions considered in a given discussion and assume that every a/h formed in it is appraisable, we are at the threshold of the classical theory. For we are in possession of an additive function obeying all the postulates required for its derivation. Its rôle is thus revealed as a theory of (unfaithful) numerical representation of relations belonging to the more far-reaching logical theory.

Before passing to the third group of results the question as to complete additivity is in order: Our axioms establish only the restricted additivity of numerical probability. The example of the infinite sequence of propositions $a_1, a_2, \cdots$ for which $a_1 \vee a_2 \vee \cdots = 1$, $a_i a_j = 0$ $(i \neq j)$ and $p(a_1/1) = p(a_2/1) = \cdots$ shows that the equation

$$1 = p(a_1 \vee a_2 \vee \cdots /1) = p(a_1/1) + p(a_2/1) + \cdots$$

is impossible.[8] This could be interpreted either by regarding the assumptions concerning $a_1, a_2, \cdots$ as self-contradictory or regarding them to be valid and holding the view that complete additivity is not a general property, but occurs only in an important class of special cases where its validity is a consequence of the physical circumstances and not of the logical aspect of probability. We have adopted the latter position.

Anyone conversant with modern theoretical physics is aware of the fundamental rôle played therein by probability. Statistical mechanics is a familiar example; but even more significant is the case of quantum mechanics, the laws of which can not be stated except in terms of probability. Now if these sciences are to be regarded as affording objective pictures of nature, how can their laws involve in an essential manner the notion of probability if this is indeed a concept of logic—a mode of thought? The answer to this question is immediate: the "probability" of these branches of physics is a misnomer for statistical weight or frequency in a sequence. If an event E in such a theory can have two possible outcomes, "success" (labeled 1) and "failure" (labeled 0), a conceptually infinite sequence of trials under "the same conditions" furnishes an infinite sequence of zeros and ones (α): $(\alpha_1, \alpha_2, \cdots)$ $(\alpha_n = 0, 1)$. The physical assumption that $w = \lim_{n\to\infty} (\alpha_1 + \cdots + \alpha_n)/n$ exists is made and this statistical weight or frequency w is what is designated by the word "probability" of success of E. But the whole objective content of the physical laws in question involves solely the notion of frequency.

Yet the intuitive conception of probability upon which the present work is based plays an essential part in connection with frequency. Its rôle becomes manifest at that very moment when the experimental significance of w is sought—significance, that is, to a pre-named individual in terms of the only phenomena which can come within his ken. Then it is that we become aware that a link is needed between the finite sets of trials—all that we can actually observe—and the mathematical idealization of frequency.[9] Analysis reveals that the only possible link is bound to involve the intuitive idea of probability.[10] Granting then the present theory, are we enabled to solve the problem? That we are indeed able to give a complete and precise solution and to do so without assuming any further principles is the content of the third group of theorems, to which we now turn.

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[9] This remains true even when frequency is thought of as a ratio in a finite sequence containing a larger number of trials than can come before the individual's observation. This is the difficulty which confronts any attempt to dispense with everything of the essence of intuitive probability and replace it by a theory of frequency. It is an attempt often made with the object of freeing the science of subjectivism (sic) and of retaining therein only an account of that which scientists "really observe"; by a strange irony it places the theory of probability completely out of contact with what any given human being could ever observe.

[10] Cf. PS, §1.

Let a_n be the experimental proposition "E succeeds at the nth trial" (so that $\alpha_n = 1$). Let h denote the statement of the common experimental condition at the instance of each trial. Then the following theorem is typical.

THEOREM. *Hypothesis* 1: $\lim_{n\to\infty} (\alpha_1 + \cdots + \alpha_n)/n = w$. *Hypothesis* 2: *For each positive integer* t

$$a_{i_1} \cdots a_{i_t}/h \approx a_{j_1} \cdots a_{j_t}/h,$$

where $(i_1, \cdots, i_t)$ *is any set of* t *distinct positive integers and* $(j_1, \cdots, j_t)$ *a similar set. Conclusion*: a_i/h *is appraisable and* $p(a_i/h) = w$.

The first idea which should enter the mind of the mathematician is that the sequence (α) can (at least when $0 < w < 1$) be reordered so as to yield a different frequency; yet apparently this must still be equal to w; is this not contradictory? The answer consists in examining the precise logical meaning of Hypothesis 1. Firstly, let $W(w, \mu, n)$ denote the assertion: *h implies that the number of true propositions in the set* $(a_1, \cdots, a_n)$ *is between* $n(w - 1/\mu)$ *and* $n(w + 1/\mu)$. Then Hypothesis 1 becomes: For any given integer μ there exists an m such that for all $n \geqq m$ assertion $W(w, \mu, n)$ is made. In logical symbols this is the familiar

$$\prod_{\mu=1}^{\infty} \sum_{m=1}^{\infty} \prod_{n=m}^{\infty} W(w, \mu, n).$$

This is all clear enough; the ambiguity appears when the logical form for assertion $W(w, \mu, n)$ is sought, for it turns out that there are many. The following is in a certain sense the weakest; it is the one for which the above theorem is proved; it is suitable for relating frequency to probability in physics:

$$W(w, \mu, n): \qquad h \subset \sum_{(p,q)} a_{p_1} \cdots a_{p_t} \sim a_{q_1} \cdots \sim a_{q_f}.$$

Here the $\sum$ calls for the disjunction of all terms where $(p_1, \cdots, p_t, q_1, \cdots, q_f)$ are all possible sets of $t+f$ distinct integers between 1 and n and where t is the least integer $\geqq n(w - 1/\mu)$ and f the least integer $\geqq n(1 - w - 1/\mu)$. But Hypothesis 1 with this form of $W(w, \mu, n)$ is not the one which makes it possible to reorder the sequence so as to produce the contradiction. For this purpose it is necessary to replace it by the entirely different $W'(w, \mu, n)$:

$$W'(w, \mu, n): \qquad \sum_{(p,q)} (h \subset a_{p_1} \cdots a_{p_t} = a_{q_1} \cdots = a_{q_f}).$$

Thus the paradox is resolved as in earlier cases by maintaining the distinction between an assertion of a disjunction and a disjunction of assertions.

The complementary rôle of the hypotheses of this theorem is worthy of note. Hypothesis 1 exhausts the purely objective state of affairs, while Hypothesis 2 which expresses a sort of intuitive random quality, contains the assumption of a subjective nature which allows an actual living being to capture the otherwise inaccessible objective fact and relate it to his own world of possible experience by the agency of intuitive probability. Thus the theorem renders unto objective reality that that is objective, and unto the subjective intuition that which pertains thereto.

No discussion of the bases of probability would be complete at the present day which did not make reference to alleged cases of the violation of certain principles of classical probability by the phenomena of quantum mechanics. The precise form at which we have here arrived makes it particularly simple to subject every such case to minute scrutiny. While we have no time for examples here, we are publishing elsewhere a discussion which shows that it is the physical circumstances to which the laws of probability apply and never the laws themselves which are altered.[11] It would indeed be hard to imagine how it could be otherwise. For insofar as the laws of probability are laws of thought, they are prior to experimental verification in the laboratory. For how indeed can such experiments prove any statement? Firstly, when the statement is an experimental proposition, then a crucial experiment suffices: but this is evidently not the case for the axioms of probability, which are not experimental propositions. Secondly, when the statement introduces harmony and intelligibility into an ensemble of statements proved in the laboratory: but the axioms of probability appear rather in the rôle of the *criteria* of such harmony and intelligibility. To argue, finally, that the axioms repose on *subjective* experiments and hence are experimental in character is beside the point since we are considering quantum mechanics which is based on experiments on electron tubes and things of this sort which are hardly in a class with the subjective experiments by means of which we become aware of our own rational processes.

Having dwelt so long on the positive side, it behooves us to mention a fundamental limitation of these results. The theory cannot prove that the probability of heads on the toss of a coin is $\frac{1}{2}$. More generally, it is as impotent to derive a non-trivial $(\prec)$ proposition from a

[11] Cf. PS, §6.

set of propositions of whatever character not containing a $(\prec)$ proposition (stated or implied) as are the laws of Newtonian mechanics to predict the position of a particle at a given time when no initial conditions are assigned. For after all, the whole theoretical structure is but the statement *in extenso* of the laws of consistency governing an aggregate of $(\prec)$ propositions.

The question is naturally raised whether some further principle of a purely formal-logical nature can be enunciated which will establish $(\prec)$ propositions *ab ovo*. Having searched high and low in the literature we become aware that every apparent case of such a principle either contains in some veiled form a $(\prec)$ proposition in its hypothesis, or else leads to insurmountable paradoxes, as in the case of the principle of sufficient reason or symmetry of ignorance which has so long sullied the name of a priori intuitive probability. The quest for the first $(\prec)$ proposition is epitomized by the attempt to devise an experiment proving the irrelevance of some external condition A in a trial of an event E. Such a statement of irrelevance is of course a $(\prec)$ proposition. In order to reason that A is irrelevant to E on one occasion from the results of experiments performed on another, one must assume that certain other unavoidable differences between the two occasions are themselves irrelevant to the situation. The difficulty, exactly contrary to Napoleon's Guard, always retreats but does not expire.

It is in the light of experience such as this that we may well ask whether it is not a principle of epistemology itself that blocks our path; and, even as those who having sought in vain for perpetual motion ended by making a virtue of their failure, so we may hazard the view that in principle the authority for the first $(\prec)$ proposition does not reside in any general law of probability, logic, or experimental science. And the notion presents itself that such primary and irreducible assumptions are grounded on a basis as much of the aesthetic as of the logical order.

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