A NOTE ON MAXIMUM MODULUS AND THE ZEROS OF AN INTEGRAL FUNCTION

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Let $f(z)$ be an integral function of finite order $\rho \geqq 0$, let $M(r, f) = M(r) = \max_{|z|=r} |f(z)|$, and let $n(r, f) = n(r)$ be the number of zeros of $f(z)$ in $|z| \leqq r$ and on its circumference. I have discussed elsewhere[1] the behaviour of $g(r) = \log M(r)/n(r)$, as $r \to \infty$, and have proved that *for every canonical product function* $f(z)$

$$(1) \qquad \liminf_{r=\infty} \frac{\log M(r)}{n(r)\phi(r)} = 0,$$

where $\phi(r)$ *is any positive L function*[2] *such that*

$$(2) \qquad \int^{\infty} \frac{dx}{x\phi(x)} < A = \text{const.}$$

The question arises how large and how small $g(r)$ and $G(r) = T(r, f)/n(r)$ can be, where $T(r, f) = T(r)$ is the Nevanlinna characteristic function for $f(z)$. I prove in this note the following result.

THEOREM. *Given* $\rho \geqq 0$ *and* $\psi(x)$ *any positive function such that*

$$\limsup_{x=\infty} \frac{\log \psi(x)}{\log x} \leqq \rho.$$

There exists an integral function $F(z)$ *of order* ρ *for which*

$$(3) \qquad \limsup_{r=\infty} \frac{T(r, F)}{\psi(r)n(r, F)} = \infty$$

and an integral function $f(z)$ *of order* ρ *for which*

$$(4) \qquad \liminf_{r=\infty} \frac{\psi(r)T(r, f)}{n(r, f)} = 0.$$

PROOF. We shall first construct an integral function $f(z)$ of order ρ for which

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[1] (i) *A theorem on integral functions of integral order*, Journal of the London Mathematical Society, vol. 15 (1940), pp. 23–31. (ii) *On integral functions of integral or zero order*, to be published. (iii) *On integral functions of perfectly regular growth*, Journal of the London Mathematical Society, vol. 14 (1939), pp. 293–302.

[2] For definition see G. H. Hardy, *Orders of Infinity*, 1924, p. 17.

$$\liminf_{r=\infty} \frac{\log M(r, f)\psi(r)}{n(r, f)} = 0 \tag{4.1}$$

from which (4) will follow. Let

$$\lambda_n = n^{n^n}, \qquad \psi(x) = e^{(\rho+\eta_x')\log x}$$

where we may suppose without loss of generality that $\eta_x' > 0$. We know that $\eta_x' \to 0$ as $x \to \infty$. Let η_r be the upper bound of η_t' for $t \geqq r$. Then $\eta_r \to 0$ monotonically as $r \to \infty$, and $\psi(r) \leqq e^{(\rho+\eta_r)\log r}$. Let $\epsilon_1 = \epsilon_2 = 1$, and

$$\epsilon_n = 2\eta_n + \frac{(4+\rho)\log \lambda_{n-1}}{\log \lambda_n}, \qquad n = 3, 4, 5, \cdots;$$

$$k = [\rho] + 1;$$

$$\mu_n = [\lambda_n^{\rho+\epsilon_n}], \quad \zeta_n = [\lambda_n^{\rho/k+\epsilon_n}], \quad n = 1, 2, 3, \cdots;$$

$$f(z) = \prod_1^\infty \left\{1 + \left(\frac{z}{\lambda_n}\right)^{k\mu_n}\right\}; \qquad F(z) = \prod_1^\infty \left\{1 + \frac{z^k}{\lambda_n}\right\}^{\zeta_n}.$$

It is easily seen that $f(z)$ and $F(z)$ are canonical products. Further, $n(\lambda_n, f) = k(\mu_1 + \cdots + \mu_n) \sim k\mu_n$ and hence

$$\limsup_{r=\infty} \frac{\log n(r, f)}{\log r} = \limsup_{n=\infty} \frac{\log k + \log \mu_n}{\log \lambda_n} = \rho.$$

Hence $f(z)$ is an integral function of order ρ. Let now $x = \lambda_m$. Then

$$\begin{aligned}\log f(x) &= \sum_{n=1}^{m-1} \log \left\{1 + \left(\frac{\lambda_m}{\lambda_n}\right)^{k\mu_n}\right\} \\ &\qquad + \log 2 + \sum_{m+1}^{\infty} \log \left\{1 + \left(\frac{\lambda_m}{\lambda_n}\right)^{k\mu_n}\right\} \\ &= \Sigma_1 + \log 2 + \Sigma_3.\end{aligned}$$

We have

$$\Sigma_3 \leqq \sum_{m+1}^{\infty} \left(\frac{\lambda_m}{\lambda_n}\right)^{k\mu_n} \leqq \sum_{m+1}^{\infty} \left(\frac{1}{2}\right)^{k\mu_n} = O(1),$$

$$\Sigma_1 < m \log \left\{2(\lambda_m)^{k\mu_{m-1}}\right\}.$$

Hence for $m \geqq m_0$,

$$\begin{aligned}\log f(x) &< 2m\{\log 2 + k\mu_{m-1} \log \lambda_m\} \\ &\leqq \exp\{\log \mu_{m-1} + 2 \log\log \lambda_m\},\end{aligned}$$

$$\frac{\psi(x)\log f(x)}{n(x,f)} \leqq \exp\left\{(\rho+\eta_{\lambda_m})\log\lambda_m+\log\mu_{m-1}+2\log\log\lambda_m-\log\mu_m\right\}$$

$$\leqq \exp\left\{(\rho+\eta_m)\log\lambda_m+\log\mu_{m-1}+2\log\log\lambda_m\right.$$
$$\left.-(\rho+2\eta_m+(4+\rho)\log\lambda_{m-1}/\log\lambda_m)\log\lambda_m+O(1)\right\},$$

and the last expression tends to zero with $1/m$. Hence

$$\liminf_{r=\infty}\frac{\psi(r)\log M(r,f)}{n(r,f)}=0$$

which proves (4.1). Further

$$n(\lambda_n^{1/k},F)=k(\zeta_1+\cdots+\zeta_n)\sim k\zeta_n.$$

Hence $F(z)$ is an integral function of order ρ. Let now $r=\frac{1}{2}\lambda_n^{1/k}$. Then

$$n(r,F)=k(\zeta_1+\cdots+\zeta_{n-1})\leqq k\zeta_{n-1}\left(1+\frac{2\zeta_{n-2}}{\zeta_{n-1}}\right)$$
$$<\exp\left\{\log k+\log\zeta_{n-1}+\frac{2\zeta_{n-2}}{\zeta_{n-1}}\right\},$$

$$\log M(\tfrac{2}{3}r,F)>\zeta_n\log\left(1+\frac{1}{3^k}\right)$$
$$=A\exp\left\{\left(\frac{\rho}{k}+\epsilon_n\right)\log\lambda_n+O(\lambda_n^{-\rho/k})\right\},$$

for $n\geqq n_0$. Hence[3]

$$\frac{\log M(\frac{2}{3}r,F)}{n(r,F)\psi(r)}>A\exp\left\{\left(\frac{\rho}{k}+\epsilon_n\right)\log\lambda_n-\log\zeta_{n-1}\right.$$
$$\left.-(\rho+\eta_n)\left(\frac{1}{k}\log\lambda_n-\log 2\right)+O(1)\right\}$$
$$=A\exp\left\{\epsilon_n\log\lambda_n-\log\zeta_{n-1}-\frac{\eta_n}{k}\log\lambda_n+O(1)\right\}$$
$$=A\exp\left\{2\eta_n\log\lambda_n+(4+\rho)\log\lambda_{n-1}\right.$$
$$\left.-\left(\frac{\rho}{k}+\epsilon_{n-1}\right)\log\lambda_{n-1}-\frac{\eta_n}{k}\log\lambda_n+O(1)\right\},$$

[3] A denotes a positive constant.

and the last expression tends to infinity with n. Hence

$$\limsup_{r=\infty} \frac{\log M(\frac{2}{3}r, F)}{n(r, F)\psi(r)} = \infty,$$

and so

$$\limsup_{r=\infty} \frac{T(r, F)}{\psi(r)n(r, F)} = \infty.$$

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