by Theorem 1 the values $a=b=c=2$ make (2) realizable. Hence there is not a unique minimal set of values for a, b, and c.

THEOREM 3. *Given I_{ij}, $i \neq j$, of* (1) *(but not the diagonal elements), let a be fixed; then we can take $I_{aa} = \max |I_{aj}|$, $j \neq a$, and find values for I_{ii}, $i \neq a$, so that* (1) *is realizable.*

The proof is similar to the proofs of the preceding theorems.

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ON THE SPHERICAL SURFACE OF SMALLEST RADIUS ENCLOSING A BOUNDED SUBSET OF n-DIMENSIONAL EUCLIDEAN SPACE[1]

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1. Introduction. An $(n-1)$-dimensional spherical surface $S_{n-1,r}$ is the "surface" of an n-dimensional sphere of radius r in E_n, the n-dimensional euclidean space. A given spherical surface encloses M, a subset of E_n, provided M is contained in the sphere with this surface, while M is enclosable by a given $S_{n-1,r}$ whenever M is a subset of a sphere whose surface is congruent with $S_{n-1,r}$. The purpose of this article is to show (1) *if M is any bounded subset of E_n (containing more than a single point) there exists a unique $S_{n-1,r}$ of smallest radius r enclosing M* and (2) *if d is the diameter of M, then the radius of the unique smallest $S_{n-1,r}$ enclosing M satisfies the relation $r \leqq [n/2(n+1)]^{1/2} \cdot d$.*

In a proof that abounds with algebraic difficulties, H. W. E. Jung established these results in his dissertation (1901) for the case of finite point sets and indicated their extension to infinite sets at the end of his long paper.[2] Returning to the subject eight years later, Jung attempted a geometric proof for the case of *n points in a plane*, but succeeded in obtaining in this later article only necessary conditions on the smallest circle enclosing a plane (finite) set, since his procedure yields the smallest circle only in case one is assured of the existence of such a circle.[3] Though this fact can readily be supplied, the geometric considerations used by Jung are not easily extended to finite

[1] Presented to the Society, February 22, 1941.

[2] H. W. E. Jung, *Ueber die kleinste Kugel, die eine räumliche Figur einschliesst*, Journal für die reine und angewandte Mathematik, vol. 123 (1901), pp. 241–257.

[3] H. W. E. Jung, *Ueber den kleinsten Kreis, der eine ebene Figur einschliesst*, ibid., vol. 137 (1909), pp. 310–313.

point sets of n-dimensional space, while for infinite sets (even of the plane) some of the argument is not valid.[4]

It seems worthwhile, therefore, to present a simple proof of the interesting assertions (1) and (2) which avoids the analytical complexities featuring Jung's demonstration and more recent proofs (see §4), deals directly with the general case of subsets of E_n in an elementary manner, and embraces in one argument both finite and infinite sets.

2. **Some lemmas.** Of the three lemmas established in this section, the first one is of interest apart from the application given it in this paper.

LEMMA 1. *If each set of $n+1$ points of a subset M of E_n is enclosable by $S_{n-1,r}$ of given radius r, then M is itself enclosable by this $S_{n-1,r}$.*

PROOF. Consider the family of spheres with centers at points of M and radius r. Since each set of $n+1$ points of M is enclosable by $S_{n-1,r}$ it is clear that each $n+1$ of these spheres have a point in common. Thus each $n+1$ of a family of convex bodies in E_n have a common point, and it follows from a theorem of Helly that there is a point p common to all the members of the family.[5] Each point of M has, then, a distance from p not exceeding r, and hence M is a subset of the sphere with radius r and center p. Thus, the surface $S_{n-1,r}$ of this sphere encloses M.

This lemma permits the reduction of the problem to a finite one concerning $n+1$ points.

LEMMA 2. *Let $P=(p_1, p_2, \cdots, p_{n+1})$ be a set of $n+1$ points of E_n with diameter $d>0$. There exists a positive number r such that P is enclosable by $S_{n-1,r}$ and not enclosable by any S_{n-1,r^*} with $r^*<r$.*

The proof follows readily from the fact that bounded subsets of the E_n are compact.

We establish now some properties of an $S_{n-1,r}$ of smallest radius enclosing an independent set P of $n+1$ points, $p_1, p_2, \cdots, p_n, p_{n+1}$, of E_n.

[4] Thus, for example, the obvious property of containing at least one point of the finite set on its circumference is not necessarily carried over for infinite sets. Again, in the treatment of the plane finite case by Rademacher and Toeplitz (*Von Zahlen und Figuren*, 2d edition, 1933, pp. 83–89), it is shown that no arc of the smallest circle as large as a semicircle can be free from points of the set. This also may be invalid when infinite sets are considered.

[5] E. Helly, *Über Mengen konvexer Körper mit gemeinschaftlichen Punkten*, Jahresbericht der Deutschen Mathematiker-Vereinigung, vol. 32 (1923), pp. 175–176. The use of Helly's theorem to prove Lemma 1, conjectured by one of us, occurs in C. V. Robinson's Missouri thesis.

PROPERTY 1. *The center c of $S_{n-1,r}$ is a point of the simplex whose vertices are the points of P.*

PROOF. If the contrary be assumed, a "face" of the simplex separates c from the vertex opposite this hyperplane. It is at once apparent that the $(n-1)$-dimensional spherical surface S_{n-1,r^*} erected on the intersection of this hyperplane with $S_{n-1,r}$ encloses P and has a radius r^* less than r.[6]

PROPERTY 2. *If a point of P is not on $S_{n-1,r}$, then c lies in the face of the simplex opposite this point.*

PROOF. Assuming the contrary, let p_j be a point of P not on $S_{n-1,r}$, and select a Cartesian coordinate system so that the $(n-1)$-dimensional hyperplane determined by the points $p_1, p_2, \cdots, p_{j-1}, p_{j+1}, \cdots, p_{n+1}$ has equation $x_n=0$ and $x_n^{(j)}>0$. It follows then from Property 1 that c_n, the nth coordinate of c, is positive. Let t be any positive number less than the smaller of the numbers $|S_{n-1,r}(p_j)|/2x_n^{(j)}$, c_n, and consider the $(n-1)$-dimensional spherical surface S_{n-1,r^*} with equation

$$S_{n-1,r} + 2tx_n = 0.$$

It is clear that the left-hand member of this equation is negative or zero for each of the points $p_1, p_2, \cdots, p_{j-1}, p_{j+1}, \cdots, p_{n+1}$ since $S_{n-1,r}$ encloses these points and each of them is in the plane $x_n=0$. But also

$$S_{n-1,r}(p_j) + 2tx_n^{(j)} < S_{n-1,r}(p_j) + |S_{n-1,r}(p_j)| = 0,$$

according to the selection of t, and hence S_{n-1,r^*} encloses all of the $n+1$ points of P. This is impossible, for since S_{n-1,r^*} is a linear combination of $S_{n-1,r}$ and the plane $x_n=0$, it passes through the intersection of these loci, and since $0<t<c_n$ the center $(c_1, c_2, \cdots, c_{n-1}, c_n-t)$ of S_{n-1,r^*} is nearer the plane $x_n=0$ than is the center c of $S_{n-1,r}$. Hence $r^*<r$ and the minimum property of r is contradicted.

Remark 1. It follows from Properties 1 and 2 that if c is an *interior* point of the simplex with vertices $p_1, p_2, \cdots, p_{n+1}$ then $S_{n-1,r}$ is the surface of the sphere circumscribing the simplex.

Remark 2. The surface of the sphere circumscribing a simplex is an $(n-1)$-dimensional spherical surface of smallest radius enclosing the

[6] In a properly selected Cartesian coordinate system the separating hyperplane has equation $x_n=0$, and the nth coordinates c_n and $x_n^{(n+1)}$ of c and p_{n+1} (the opposite vertex), respectively, differ in sign. S_{n-1,r^*} has equation $S_{n-1,r}+2c_nx_n=0$, and hence evidently encloses the $n+1$ points $p_1, p_2, \cdots, p_{n+1}$, while $r^*<r$ since the center of S_{n-1,r^*} is the orthogonal projection of c on $x_n=0$.

vertices of the simplex if and only if the circumcenter is a point of the simplex. Since we shall not use this simple criterion in establishing the general theorem, we omit its proof.

LEMMA 3. *Let P be a set of $n+1$ points of E_n, not of E_{n-1}, of diameter d. If $S_{n-1,r}$ is an $(n-1)$-dimensional spherical surface of smallest radius r enclosing P, then $r \leqq [n/2(n+1)]^{1/2} \cdot d$.*

PROOF. Let $P_1, P_2, \cdots, P_{n+1}$ be vectors corresponding to the $n+1$ points $p_1, p_2, \cdots, p_{n+1}$ of P, respectively, while C denotes the vector corresponding to the center c of $S_{n-1,r}$. From Property 1, c is a point of the simplex whose vertices are the points of P and hence *non-negative* constants $k_1, k_2, \cdots, k_{n+1}$ exist such that

$$C = k_1P_1 + k_2P_2 + \cdots + k_{n+1}P_{n+1}, \tag{1}$$

with $k_1+k_2+\cdots+k_{n+1}=1$, and we may suppose the labeling of the $n+1$ points so that (2) $k_{n+1} \geqq k_i$ $(i=1, 2, \cdots, n+1)$. Then k_{n+1} is surely positive and hence c does not lie in the face of the simplex opposite p_{n+1}. It follows from Property 2 that p_{n+1} is on $S_{n-1,r}$.

Translating the origin of coordinates to p_{n+1} does not change the constants $k_1, k_2, \cdots, k_{n+1}$, and we have the scalar product $P_i \cdot (2C-P_i)$ of the vectors P_i and $2C-P_i$ equal to zero *for each index i such that* $k_i>0$, for if k_i is positive then p_i is on $S_{n-1,r}$. Hence, for each such index i, $2P_i \cdot C = P_i \cdot P_i$, but the equality $2k_iP_i \cdot C = k_i(P_i \cdot P_i)$ evidently holds for *every* $i=1, 2, \cdots, n+1$, since k_i is positive or zero. Summing for $i=1, 2, \cdots, n$, we get

$$\sum_{i=1}^{n} (k_iP_i) \cdot C = \frac{1}{2} \sum_{i=1}^{n} k_i(P_i \cdot P_i) = \frac{1}{2} \sum_{i=1}^{n} k_id_i^2,$$

where d_i denotes the length of P_i. Using (1) and recalling that the diameter of P is d, we conclude (since $P_{n+1}=0$) that $C \cdot C \leqq \frac{1}{2}d^2\sum_{i=1}^{n}k_i$, from which it follows at once (since $\sum_{i=1}^{n}k_i = 1-k_{n+1}$ and using (2)) that $r^2 \leqq [n/2(n+1)] \cdot d^2$; that is, $r \leqq [n/2(n+1)]^{1/2} \cdot d$, and the lemma is proved.[7]

Apart from the uniqueness (which is proved in the general theorem that follows), the Lemmas 2 and 3 prove our theorem for simplices in E_n, while Properties 1 and 2 and the accompanying remarks give important characteristics of a smallest enclosing $S_{n-1,r}$. Lemma 1 presents the means of extending the proof to any bounded subset of E_n.

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[7] The treatment by vectors was suggested by C. Herring as a more elegant presentation than the one originally obtained.

3. **The theorem.** We are now in a position to prove quite easily the principal theorem.

THEOREM. *Let d be the diameter of the bounded set M (containing more than a single point) of the n-dimensional euclidean space E_n. Then* (1) *there exists a unique smallest spherical surface $S_{n-1,r}$ enclosing M and* (2) $r\leqq [n/2(n+1)]^{1/2}\cdot d$.

PROOF. The theorem is obviously valid for $n=1$ (the desired $S_{0,r}$ evidently consists of the endpoints of the closure of M). We make the inductive hypothesis of its validity for every positive integer k less than n.

Case 1. *M is a subset of E_k, $1\leqq k<n$.* Then by the inductive hypothesis there exists a unique smallest $S_{k-1,r}$ enclosing M and $r\leqq [k/2(k+1)]^{1/2}\cdot d<[n/2(n+1)]^{1/2}\cdot d$ since $k<n$. It is clear that the $S_{n-1,r}$ which is the surface of the n-dimensional sphere whose radius is r and whose center coincides with that of the k-dimensional sphere with surface $S_{k-1,r}$ satisfies the requirements of the theorem.

Case 2. *M is not a subset of E_k, $k<n$.* Then the set $\{P\}$ of all sets of $n+1$ points of M is not empty, and by Lemma 2 there is a smallest $S_{n-1,r(P)}$ enclosing each P of $\{P\}$. Define $r=\text{l.u.b.}_{P\epsilon\{P\}}r(P)$. Since $0<r(P)<d$, $P\in\{P\}$, r is a positive (finite) number.

ASSERTION. *M is enclosable by $S_{n-1,r}$ and by no spherical surface of smaller radius.*

First, since $r\geqq r(P)$, $P\subset M$, it follows that each set of $n+1$ points of M is enclosable by $S_{n-1,r}$ and hence, by Lemma 1, M is itself enclosable by $S_{n-1,r}$. Second, an assumption that M is enclosable by S_{n-1,r^*}, $r^*<r$, leads at once (according to the definition of r) to the existence of a subset P of $n+1$ points of M with $r(P)>r^*$; that is, the smallest spherical surface enclosing this subset P has a radius exceeding r^*. Hence this subset (and consequently M) is not enclosable by an S_{n-1,r^*}.

Let $S_{n-1,r}(p)$ denote an $(n-1)$-dimensional spherical surface of smallest radius r enclosing M with center p. If, now, $S_{n-1,r}(q)$ is another such spherical surface, then M is contained in the common part of the two n-dimensional spheres of radius r and centers p and q, and consequently is part of an n-dimensional sphere of radius $r^*<r$. Then M is enclosable by S_{n-1,r^*}, $r^*<r$, which is impossible. Hence the center as well as the radius of $S_{n-1,r}$ is fixed and the uniqueness is established.

Finally, $r(P)\leqq [n/2(n+1)]^{1/2}\cdot d$ for each P of $\{P\}$. For if P is in E_k, $k<n$, this follows from Case 1, while if P consists of $n+1$

independent points then the result follows from Lemma 3. Hence $r = \text{l.u.b.}_{P\epsilon\{P\}} r(P) \leqq [n/2(n+1)]^{1/2}\cdot d$, and the proof of the theorem is complete.

If $p_1, p_2, \cdots, p_{n+1}$ are the vertices of an equilateral simplex of edge d then the unique $(n-1)$-dimensional spherical surface of smallest radius enclosing the points is, according to Remark 2, the surface of the circumscribed sphere. Since for this sphere $r = [n/2(n+1)]^{1/2}\cdot d$, the inequality proved for r in the theorem cannot be sharpened.

4. **Some historical remarks.** Though Jung does not mention the fact in his dissertation, some aspects of the problem he dealt with had been considered long before. Thus Sylvester, in a paper on approximate valuation of surd forms, asserted that the essential preliminary question to be resolved was that of cutting off by a plane the smallest possible segment of a sphere that should contain the whole of a given (finite) set of points lying on the surface.[8] He stated that some years earlier he had proposed the problem of drawing the smallest circle enclosing a given finite set of points in the plane "without any suspicion of its having a practical application," and that "by a singular coincidence, Professor Peirce, of Cambridge University (*sic*), U.S., has studied this question and obtained a complete solution" which was applicable to the analogous problem on the sphere.

Unaware of either Sylvester's or Peirce's connection with the problem, G. Chrystal (1884) gave a geometrical algorithm for drawing a smallest circle enclosing a given finite set of points in the plane.[9] No mention is made in any of this earlier work of a relationship between the diameter of the set and the radius of the smallest circle containing it.

In 1914 M. Bricard, apparently unaware of either of Jung's two earlier papers, proved that every closed plane curve of elongation (diameter) d is contained in a circle of radius $d/3^{1/2}$ and every closed surface of diameter d is contained in a sphere of radius $(3/8)^{1/2}d$.[10] In a brief note H. Lebesgue remarked that Bricard's methods might be applied to any plane set E of diameter d by considering E as a subset of the set E^* which is *saturated with respect to having diameter*

[8] J. J. Sylvester, *On Poncelet's approximate valuation of surd forms*, Philosophical Magazine, vol. 20 (1860), pp. 203–222; see also *Collected works*, vol. 2, pp. 118, 183.

[9] G. Chrystal, *On the problem to construct the minimum circle enclosing n given points in a plane*, Proceedings of the Edinburg Mathematical Society, vol. 3 (1884–1885), pp. 30–33.

[10] M. Bricard, *Théorèmes sur les courbes et les surfaces fermées*, Nouvelles Annales de Mathématiques, (4), vol. 14 (1914), pp. 19–25.

d.[11] Such sets E^* are convex domains bounded by curves that Euler called *orbiform* (that is, curves of constant breadth).

A kind of dual of Jung's theorem was proved by W. Blaschke (1914).[12] If E is a plane convex set, the breadth of E is defined as the minimum distance of two parallel supporting lines of E. Blaschke showed that *the greatest circle which is contained in every plane convex set of breadth* 1 *has diameter* $2/3$, and established analogous results for higher dimensions. It was pointed out by J. v. Sz. Nagy that Blaschke's methods could be applied to prove Jung's theorem.[13] In 1917 K. Reinhardt sought to extend to the E_n the geometrical arguments used by Jung in 1909 for subsets of E_2.[14] The demonstration is tedious and not entirely non-intuitive. An analogue of Jung's theorem in more general spaces was proved in 1938 by F. Bohnenblust.[15]

Finally, we note that in 1905 E. Landau applied Jung's theorem in the plane to sharpen an inequality in the theory of analytic functions due to F. Schottky.[16]

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[11] H. Lebesgue, *Sur les courbes orbiformes, à propos d'une note récente de M. R. Bricard*, Bulletin de la Société Mathématique de France, Comptes Rendus des Séances, 1914, pp. 45–46 (abstract).

[12] W. Blaschke, *Über den grössten Kreis in einer konvexen Punktmenge*, Jahresbericht der Deutschen Mathematiker-Vereinigung, vol. 23 (1914), pp. 369–374.

[13] J. v. Sz. Nagy, *Über einen Satz von H. Jung*, ibid., vol. 24 (1915), pp. 390–392

[14] K. Reinhardt, *Über die kleinste Kugel, die um jede Punktmenge vom Durchmesser Eins gelegt werden kann*, ibid., vol. 25 (1917), pp. 157–163.

[15] F. Bohnenblust, *Convex regions and projections in Minkowski spaces*, Annals of Mathematics, (2), vol. 39 (1938), pp. 301–308.

[16] E. Landau, *Über einige Ungleichheitsbeziehungen in der Theorie der analytischen Funktionen*, Archiv der Mathematik und Physik, (3), vol. 11 (1905), pp. 31–36.