be the orthonormal trigonometric sums on D_1 for weight $\rho(\theta)\sigma(\theta)$; if the u's and v's are uniformly bounded on a point set D_2 contained in D_1, the same is true of the U's and V's.

For this case the proof admits a materially simpler formulation than when geometric configurations are contemplated having the degree of generality previously considered. The details relating to the loci C', C'', K, K', K'', can be dispensed with for the most part; with θ replacing the pair of coordinates (x, y), and ϕ replacing the pair (u, v), it is sufficient, for any particular value of θ, to consider separately the intervals $(\theta-\pi/2, \theta+\pi/2)$ and $(\theta+\pi/2, \theta+3\pi/2)$, and in the integral corresponding to the right-hand member of (5) to represent $K_{n-1}(\theta, \phi)$ in the former interval by an expression with denominator $\sin(\theta-\phi)$, and in the latter interval by an alternative expression with $1-\cos(\theta-\phi)$ in the denominator.[10]

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[10] [B, pp. 808–809.]

APPROXIMATION OF CONTINUOUS FUNCTIONS BY MEANS OF LACUNARY POLYNOMIALS

BERNARD DIMSDALE

1. Introduction. All rational integral polynomials are linear combinations of members of the complete set of powers whose exponents are the non-negative integers. If certain members of this set are deleted, the linear combinations formed from the resulting set are, in the strict sense of the term, "lacunary polynomials." In a large part of this paper, however, methods of reasoning designed for the treatment of such polynomials are applicable to combinations from much more general sets of powers whose exponents are non-negative but not in general integral. The term "polynomial in x^{μ} of degree μ_n" will be applied to combinations from the set $1, x^{\mu_1}, x^{\mu_2}, \cdots$ where $\mu_1, \mu_2, \cdots$ form an arbitrarily preassigned set of real numbers such that $0<\mu_1<\mu_2<\cdots$, and μ_n is the largest exponent.

This paper started out as an investigation of lacunary orthogonal polynomials, and although this aspect of it became subordinate to the

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problem of approximation, the relation of the latter to the theory of convergence of series of the orthogonal polynomials is pointed out at the end.

2. **Preliminary results.** It has been shown[1] that for all positive integers m the trigonometric sum

$$I_m(t) = h_m \int_{-\pi/2}^{\pi/2} g(t + 2u) F_m(u) du$$

with

$$F_m(u) = \left[\frac{\sin mu}{m \sin u}\right]^4, \qquad \frac{1}{h_m} = \int_{-\pi/2}^{\pi/2} F_m(u) du,$$

of order $2m-2$, satisfies $|I_m(t) - g(t)| \leqq D\lambda/m$, for all t, provided that $g(t)$ has period 2π and satisfies a Lipschitz condition, constant λ, for all real t, where D is an absolute constant; and, furthermore, that if $g(t)$ is an even function, $I_m(t)$ will be a cosine sum, $I_m(t) = \alpha_0^{(m)} + \alpha_1^{(m)} \cos t + \cdots + \alpha_{2m-2}^{(m)} \cos (2m-2)t$.

Now

$$\alpha_k^{(m)} = \frac{1}{\pi} \int_0^{2\pi} I_m(t) \cos kt\, dt,$$

and when $I_m(t)$ is replaced by its integral form, order of integration reversed, and the substitution $t + 2u = y$ made,

$$\alpha_k^{(m)} = \frac{h_m}{\pi} \int_{-\pi/2}^{\pi/2} F_m(u) \left[\int_0^{2\pi} \cos k(y - 2u) g(y) dy\right] du,$$

since $g(t)$ has period 2π. Since[2]

$$\left| \frac{1}{\pi} \int_0^{2\pi} g(t) \begin{matrix}\cos \\ \sin\end{matrix}\, kt\, dt \right| \leqq \frac{\pi\lambda}{k}$$

it follows on expanding $\cos k(y-2u)$ that[3]

$$\left|\alpha_k^{(m)}\right| \leqq \frac{h_m}{\pi} \int_{-\pi/2}^{\pi/2} F_m(u) \frac{\pi^2\lambda}{k} du + \frac{h_m}{\pi} \int_{-\pi/2}^{\pi/2} F_m(u) \frac{\pi^2\lambda}{k} du,$$

$$\left|\alpha_k^{(m)}\right| \leqq \frac{2\pi\lambda}{k}, \qquad k = 1, 2, \cdots, m-1; m = 2, 3, \cdots.$$

[1] D. Jackson, *The Theory of Approximation*, American Mathematical Society Colloquium Publications, vol. 11, 1930, pp. 2–6.

[2] See, for example, Titchmarsh, *The Theory of Functions*, p. 426.

[3] The following shorter proof has been given $\left|\alpha_k^{(m)}\right| = \left|(1/\pi)\int_0^{2\pi}[I_m(t) - g(t) + g(t)] \cos kt\, dt\right| \leqq 2D\lambda/m + \pi\lambda/k < \lambda(2D+\pi)/k$. The different value of the constant here would change subsequent computations slightly.

If $g(x)$ satisfies a Lipschitz condition, constant λ, on the closed interval[4] $[-1, 1]$, then for each positive integer m there exists a polynomial $P_m(y)$ in $1, y, y^2, \cdots$, of degree $2m-2$, such that on $[-1, 1]$

$$| g(y) - P_m(y) | \leqq \frac{D\lambda}{m} \cdot$$

Here[5] $P_m(y) = I_m(t)$, where $I_m(t)$ is defined for $g(\cos t)$ and $y = \cos t$. Let $P_m(y) = a_0^{(m)} + a_1^{(m)} y + \cdots + a_{2m-2}^{(m)} y^{2m-2}$. The following inequalities result directly from the fact that $P_m(y) = I_m(t)$;

$$| a_{2k}^{(m)} | \leqq \frac{2^{2k}\pi\lambda}{2k} C_{2k}(m + k - 1),$$

$$| a_{2k-1}^{(m)} | \leqq \frac{2^{2k-1}\pi\lambda}{2k - 1} C_{2k-1}(m + k - 2),$$

where

$$C_r(s) = \frac{s!}{(s - r)!r!}, \qquad k = 1, 2, \cdots, m - 1; m = 2, 3, \cdots .$$

Let $y = x^H$, $g(x^H) = f(x)$, $H > 0$. Then[6] $|f(x_2) - f(x_1)| \leqq \lambda |x_2^H - x_1^H|$ on $[0, 1]$, implies that for every positive integer m there exists a polynomial

$$P_{mH}(x) = a_0^{(m)} + a_1^{(m)} x^H + \cdots + a_{2m-2}^{(m)} x^{(2m-2)H}$$

such that on $[0, 1]$,

$$| f(x) - P_{mH}(x) | \leqq \frac{D\lambda}{m} \cdot$$

It will be observed that in this transformation the coefficients $a_k^{(m)}$ remain unchanged.

It has been shown by Szász[7] that for a sequence $\gamma_0, \gamma_1, \gamma_2, \cdots, \gamma_n$, where $0 < \gamma_0$, $0 < \gamma_1 < \gamma_2 < \cdots < \gamma_n$ and $\gamma_0 \neq \gamma_i$, $i = 1, 2, \cdots, n$, there exists a set of constants $u_1, u_2, \cdots, u_n$ such that

[4] Hereafter the bracket notation for an interval implies closure at both ends.

[5] See the argument in D. Jackson. loc. cit., pp. 13–14.

[6] Hereafter this condition will be called a "Lipschitz condition H." It has been pointed out that this condition implies that $f(x)$ satisfies a Lipschitz condition of order α, $\alpha = H$ if $H \leqq 1$, $\alpha = 1$ if $H \geqq 1$. However, the converse is not true, since $x = 0$ belongs to the interval in question. Hence this condition is more restrictive than a Lipschitz condition of order α. On any closed interval not including 0 the two conditions are, however, equivalent.

[7] O. Szász, *Über die Approximation Stetiger Funktionen durch lineare Aggregate von Potenzen*, Mathematische Annalen, vol. 77 (1916), pp. 485–487.

$$\int_0^1 [x^{\gamma_1-1/2} + u_1 x^{\gamma_2-1/2} + \cdots + u_n x^{\gamma_n-1/2}]^2 dx \leqq \frac{1}{2\gamma_0} \prod_{h=1}^{n} \left[\frac{\gamma_h - \gamma_0}{\gamma_h + \gamma_0}\right]^2.$$

Let

$$A(x) = \frac{x^{\gamma_0+1/2}}{\gamma_0 + 1/2} + \frac{u_1 x^{\gamma_1+1/2}}{\gamma_1 + 1/2} + \cdots + \frac{u_n x^{\gamma_n+1/2}}{\gamma_n + 1/2},$$

$$B(x) = \frac{(\gamma_0 + 1/2)A(x)}{x^{1/2}} = x^{\gamma_0} + c_1 x^{\gamma_1} + \cdots + c_n x^{\gamma_n},$$

$$C(x) = x^{\gamma_0-1/2} + u_1 x^{\gamma_1-1/2} + \cdots + u_n x^{\gamma_n-1/2}.$$

Then

$$A(x) = \int_0^x C(x)dx,$$

and

$$[A(x)]^2 = \left[\int_0^x C(x)dx\right]^2 \leqq \int_0^x dx \cdot \int_0^x [C(x)]^2 dx$$

$$= x \cdot \int_0^x [C(x)]^2 dx \leqq x \cdot \int_0^1 [C(x)]^2 dx$$

$$\leqq \frac{x}{2\gamma_0} \prod_{h=1}^{n} \left[\frac{\gamma_h - \gamma_0}{\gamma_h + \gamma_0}\right]^2 \qquad \text{on } [0, 1].$$

Therefore on $[0, 1]$

$$|A(x)| \leqq \frac{x^{1/2}}{(2\gamma_0)^{1/2}} \prod_{h=1}^{n} \frac{|\gamma_h - \gamma_0|}{\gamma_h + \gamma_0},$$

$$|B(x)| \leqq \frac{2\gamma_0 + 1}{2(2\gamma_0)^{1/2}} \prod_{h=1}^{n} \frac{|\gamma_h - \gamma_0|}{\gamma_h + \gamma_0}.$$

If $\gamma_0 = \gamma_i$ for some i, then the u's, and hence the c's, may obviously be chosen so that $|B(x)| \equiv 0$, and the inequality still holds.

It may now be shown that if $\gamma_0 = rH$, $(i-1)H < \gamma_i \leqq iH$, then

$$L_{r,n}(x) = |B(x)| \leqq \frac{2rH + 1}{2(2rH)^{1/2}} \cdot \frac{1}{C_r(2r - 1) \cdot C_{2r}(n + r)} \qquad \text{on } [0, 1],$$

for $1 \leqq r \leqq n$. The proof depends upon the fact that for $\gamma_0 \geqq \gamma_h$,

$$\frac{\gamma_0 - \gamma_h}{\gamma_0 + \gamma_h} \leqq \frac{rH - (h-1)H}{rH + (h-1)H} = \frac{r - h + 1}{r + h - 1},$$

with a similar situation for $\gamma_0 \leqq \gamma_h$. If, in particular $\gamma_i = iH$ for some i, $\gamma_0 = iH$, then $L_{r,n}(x) \equiv 0$, and the inequality is still true.

3. **Degree of approximation.** This theorem will now be proved.

THEOREM I. *Consider the sequence* $\mu_0, \mu_1, \mu_2, \cdots$, *for which* $0 = \mu_0 < \mu_1 < \mu_2 < \cdots$, *where* $\mu_{n+1} - \mu_n \leqq H$ *for all* n, *and* $\lim_{n=\infty} \mu_n = \infty$. *If* $f(x)$ *satisfies a Lipschitz condition* H, *constant* λ, *on* $[0, 1]$, *then for every integer* $n > 0$ *there exists* $P_{\mu_n}(x)$, *a polynomial in* x^μ *of degree* μ_n, *such that on* $[0, 1]$ $|f(x) - P_{\mu_n}(x)| \leqq K\lambda/\mu_n$, *where* K *depends on* H *only.*

To prove this theorem, a polynomial $P_{mH}(x) = a_0^{(m)} + a_1^{(m)} x^H + \cdots + a_{2m-2}^{(m)} x^{(2m-2)H}$ is constructed so that on $[0, 1]$ $|f(x) - P_{mH}(x)| \leqq D\lambda/m$, with bounds on the a's as given in the last section. Unless $rH = \gamma_k$, each x^{rH} is then approximated by a polynomial $P^{(r)}_{\gamma_{4m-4}}(x) = -c_1 x^{\gamma_1} - \cdots - c_{4m-4} x^{\gamma_{4m-4}}$, where the set of γ's is a subsequence of the set of μ's, $\gamma_0 = \mu_0$, γ_i is the smallest μ_n such that $(i-1)H < \gamma_i \leqq iH$, $i \geqq 1$; and $|x^{rH} - P^{(r)}_{\gamma_{4m-4}}(x)| = L_{r,4m-4}(x)$. Let

$$P_{\gamma_{4m-4}}(x) = a_0^{(m)} + \sum_{h=1}^{2m-2} a_h^{(m)} P^{(h)}_{\gamma_{4m-4}}(x).$$

Then

$$\begin{aligned} |f(x) - P_{\gamma_{4m-4}}(x)| &\leqq \frac{D\lambda}{m}, && \text{if } m = 1, \\ &\leqq \frac{D\lambda}{m} + \sum_{k=1}^{m-1} P_k + \sum_{k=1}^{m-1} Q_k, && \text{if } m = 2, 3, \cdots, \end{aligned}$$

where P_k and Q_k are the respective bounds of $|a_{2k}^{(m)}| \cdot |L_{2k,4m-4}(x)|$ and $|a_{2k-1}^{(m)}| \cdot |L_{2k-1,4m-4}(x)|$, as previously obtained. On calculation it is found that

$$\begin{aligned} \frac{P_{k+1}}{P_k} = \frac{4(k+1)H+1}{4kH+1} \cdot \frac{k^{1/2}}{(k+1)^{1/2}} \cdot \frac{2k+1}{4m+2k-3} \\ \cdot \frac{m+k}{2m+k-1} \cdot \frac{2k}{2m-k-2} \cdot \frac{m-k-1}{4m-2k-5}, \\ \frac{Q_{k+1}}{Q_k} = \frac{2(2k+1)H+1}{2(2k-1)H+1} \cdot \frac{(2k-1)^{1/2}}{(2k+1)^{1/2}} \cdot \frac{2k}{4m+2k-3} \\ \cdot \frac{m+k-1}{2m+k-2} \cdot \frac{2k-1}{2m-k-2} \cdot \frac{m-k-1}{4m-2k-3}, \end{aligned}$$

for $k = 1, 2, \cdots, m-2$; $m = 3, 4, 5, \cdots$. If now each fraction is replaced by its maximum for the given range of k with m fixed, and then by its maximum for the given range of m, it follows that, since $(8H+1)/(4H+1)\leqq 2$, $(6H+1)/(2H+1)\leqq 3$, $P_{k+1}/P_k\leqq 2/9$, $Q_{k+1}/Q_k \leqq 1/3$; hence $P_k\leqq(2/9)^{k-1}P_1$, $Q_k\leqq(1/3)^{k-1}Q_1$. Finally

$$\left| f(x) - P_{\gamma_{4m-4}}(x) \right| \leqq \frac{D\lambda}{m} + \frac{9}{7} P_1 + \frac{3}{2} Q_1, \qquad m = 1, 2, \cdots .$$

Now direct calculation shows that

$$P_1 \leqq \frac{(4H+1)\pi\lambda}{45H^{1/2}m}, \qquad Q_1 \leqq \frac{(2H+1)\pi\lambda}{5(2H)^{1/2}m}, \qquad m = 2, 3, \cdots ,$$

the equalities holding for $m=2$. Hence it appears that

$$\left| f(x) - P_{\gamma_{4m-4}}(x) \right| \leqq \frac{L\lambda}{m}, \qquad m = 1, 2, 3, \cdots ,$$

where L depends on H only.

Now let n be an arbitrary integer, and let $m=(n+4)/4$, $(n+3)/4$, $(n+2)/4$, $(n+1)/4$, respectively, according as n has the form $4k$, $4k-1$, $4k-2$, $4k-3$. In any event $4m-4\leqq n<4m$, $\gamma_n\leqq nH$. Let the corresponding $P_{\gamma_{4m-4}}(x)$ be denoted by $P_{\gamma_n}(x)$, a polynomial of degree γ_n. Since $1/m\leqq 4/n\leqq 4H/\gamma_n$ it follows that on $[0, 1]$

$$\left| f(x) - P_{\gamma_n}(x) \right| \leqq \frac{4HL\lambda}{\gamma_n} \cdot$$

Consider the original sequence x^{μ_0}, x^{μ_1}, x^{μ_2}, $\cdots$. Let $P_{\mu_n}(x)$ be a polynomial of degree μ_n formed from this sequence, $P_{\mu_n}(x)=P_{\gamma_k}(x)$ where γ_k is the largest of the γ's in $\mu_0, \mu_1, \cdots, \mu_n$. Now $\mu_n\leqq kH$, since otherwise γ_k would not be the largest γ in the given set of μ's. If $k\geqq 2$, then $2\gamma_k>(2k-2)H\geqq kH\geqq\mu_n$, $1/\gamma_k<2/\mu_n$, and $\left|f(x)-P_{\mu_n}(x)\right| \leqq 8HL\lambda/\mu_n$. If $k=0$, 1 then $P_{\mu_n}(x)=P_{\gamma_0}(x)$, $\mu_n\leqq H$, $1\leqq H/\mu_n$, and $\left|f(x)-P_{\mu_n}(x)\right|\leqq L\lambda/1\leqq HL\lambda/\mu_n$. Hence the first of these two inequalities on $f(x)$ serves for all n, and the theorem is established.

This result may be extended to $[0, b]$, $b>0$, by assuming that $f(x)$ satisfies a Lipschitz condition H, constant λ, on $[0, b]$ and substituting $x=by$. Then there exists $\overline{P}_{\mu_n}(y)$ of degree μ_n such that on $[0, 1]$

$$\left| f(by) - \overline{P}_{\mu_n}(y) \right| \leqq \frac{Kb^H\lambda}{\mu_n},$$

whence on $[0, b]$

$$\left| f(x) - P_{\mu_n}(x) \right| \leqq \frac{Kb^H\lambda}{\mu_n},$$

with $P_{\mu_n}(x)=\overline{P}_{\mu_n}(y)$.

The result may also be extended to $[a, b]$, where $a>0$, by defining $f(x)\equiv f(a)$ on $[0, a]$. In this case the condition on $f(x)$ may be replaced by one of the two simpler conditions.[6]

THEOREM II. *Suppose $f(x)$ is continuous on $[0, 1]$ and let*[8] $\omega_H(\delta)=\max|f(x_2)-f(x_1)|$ *for all x_1, x_2 on $[0, 1]$ satisfying the condition $|x_2^H-x_1^H|\leqq\delta$, and $\omega_H(\delta)\equiv\omega_H(1)$ if $\delta>1$. Then for each $\mu_n>0$ there exists a polynomial $P_{\mu_n}(x)$ such that on $[0, 1]$*

$$|f(x)-P_{\mu_n}(x)|\leqq \overline{K}\omega_H\left(\frac{1}{\mu_n}\right)$$

with $\overline{K}=K+2$.

Let

$$\phi(x)=a_ix^H+b_i,\qquad \left(\frac{i}{\mu_n}\right)^{1/H}\leqq x\leqq\left(\frac{i+1}{\mu_n}\right)^{1/H},\quad i=0,1,\cdots,$$

$$a_i=\mu_n\left\{f_1\left[\left(\frac{i+1}{\mu_n}\right)^{1/H}\right]-f_1\left[\left(\frac{i}{\mu_n}\right)^{1/H}\right]\right\},$$

$$b_i=(i+1)f_1\left[\left(\frac{i}{\mu_n}\right)^{1/H}\right]-if_1\left[\left(\frac{i+1}{\mu_n}\right)^{1/H}\right],$$

where $f_1(x)\equiv f(x)$, $[0, 1]$, $f_1(x)\equiv f(1)$, $x\geqq 1$. Then

$$|\phi(x_2)-\phi(x_1)|\leqq\mu_n\omega_H\left(\frac{1}{\mu_n}\right)|x_2^H-x_1^H|,\qquad [0,1],$$

$$|f(x)-\phi(x)|\leqq 2\omega_H\left(\frac{1}{\mu_n}\right),\qquad [0,1].$$

It follows from the first of these two inequalities that a $P_{\mu_n}(x)$ exists such that $|\phi(x)-P_{\mu_n}(x)|\leqq K\omega_H(1/\mu_n)$ on $[0, 1]$, and the theorem follows. This theorem may also be extended to $[a, b]$, $0\leqq a\leqq b$, in which case

$$|f(x)-P_{\mu_n}(x)|\leqq K'\omega_H\left(\frac{b^H-a^H}{\mu_n}\right).$$

THEOREM III. *Let μ_n be given as before and let $\mu_\alpha=H$ be the smallest of the μ's for which $\mu_{n+1}-\mu_n\leqq\mu_\alpha$ for all n. Suppose $f(x)$ has a derivative for every x on $[a, b]$, and that $x^{1-H}f'(x)$ is continuous on $[a, b]$. Let $\eta_r=\mu_{r+\alpha}-H$ for all $r\geqq 0$. Suppose that there is given a sequence of positive numbers $\{\epsilon_n\}$, and suppose that for each ϵ_n there exists $P_n(x)$, a*

[8] Hereafter the function $\omega_H(\delta)$ will be called a "modulus of continuity H."

polynomial in x^n *of degree* $\eta_{n-\alpha}$, *such that on* $[a, b]$ $|x^{1-H}f'(x)-P_n(x)| \leqq \epsilon_n$. *Then, for every* $n>\alpha$ *there exists* $P_{\mu_n}(x)$, *a polynomial in* x^μ *of degree* μ_n, *such that on* $[a, b]$

$$|f(x) - P_{\mu_n}(x)| \leqq \frac{K''\epsilon_n}{\mu_n}$$

where K'' *depends on* H, a, *and* b.

Let

$$R_n(x) = \int_a^x x^{H-1}P_n(x)dx,$$

and let $r_n(x) = f(x) - R_n(x)$. Then, with the possible exception of $x = a$, $r_n'(x)$ may be found; hence $x^{1-H}r_n'(x)$, from which it follows that $|r_n'(x)| \leqq x^{H-1}\epsilon_n$. Since

$$r_n(x_2) - r_n(x_1) = \int_{x_1}^{x_2} r_n'(x)dx,$$

it follows that on $[a, b]$

$$|r_n(x_2) - r_n(x_1)| \leqq \frac{\epsilon_n}{H}|x_2^H - x_1^H|.$$

Therefore there exists $\overline{R}_n(x)$ in x^μ of degree μ_n which approximates $r_n(x)$ with error at most $K\epsilon_n/H\mu_n$ and, with $P_{\mu_n}(x) = R_n(x) + \overline{R}_n(x)$,

$$|f(x) - P_{\mu_n}(x)| \leqq \frac{K\epsilon_n}{H\mu_n}.$$

From Theorem III it follows that if $x^{1-H}f'(x)$ satisfies a Lipschitz condition H, constant λ, on $[a, b]$, then for every $n>\alpha$ there exists $P_{\mu_n}(x)$ such that

$$|f(x) - P_{\mu_n}(x)| \leqq \frac{K_1\lambda}{\mu_n(\mu_n - H)}.$$

If $x^{1-H}f'(x)$ has a modulus of continuity H on $[a, b]$, then for every $n>\alpha$ there exists a polynomial $P_{\mu_n}(x)$ of degree μ_n for which

$$|f(x) - P_{\mu_n}(x)| \leqq \frac{K_2}{\mu_n}\omega_H\left(\frac{b^H - a^H}{\mu_n - H}\right).$$

4. **Uniform convergence.** With the sequence $\{x^{\mu_i}\}$ and a non-negative function $\rho(x)$, integrable on $[a, b]$ with positive integral, poly-

nomials $p_{\mu_n}(x)$ in x^μ of degree μ_n can be constructed for every μ_n by well known methods so that these polynomials will be orthonormal with respect to $\rho(x)$ on $[a, b]$. If $f(x)$ is an arbitrary continuous function, it follows as a formal result

$$(1) \qquad f(x) \sim \sum_{n=0}^{\infty} a_n p_{\mu_n}(x),$$

where

$$a_n = \int_a^b f(t) p_{\mu_n}(t) \rho(t) dt.$$

In the following the results of the previous section are combined with certain results from the theory of orthogonal polynomials to give theorems on convergence for the above series expansion.[9] It will be necessary at this point to suppose that the μ's are integers, since Bernstein's theorem is implicitly involved in the following. It is to be observed that this restriction admits sequences so irregular that the results obtained are by no means trivial, and further that the admission of any set of commensurable exponents would not be a material generalization.

In the following statement of those results from orthogonal polynomial theory[10] which are applied to the present problem of convergence, all polynomials are polynomials in x^μ, of degree given by their subscript, and the use of ϵ_n implies that there exists a polynomial $P_{\mu_n}(x)$ such that on $[a, b]$, $|f(x) - P_{\mu_n}(x)| \leqq \epsilon_n$. If $\overline{Q}_{\mu_n}(x)$ is a polynomial minimizing[11]

$$\int_a^b \rho(x)[f(x) - Q_{\mu_n}(x)]^2 dx,$$

if $\rho(x)$ has a positive lower bound on $[a, b]$, then there are constants C_1, C_2, independent of x and n, such that

[9] For the sequence $1, x^{\mu_1}, \cdots, x^{\mu_{N-1}}; x^\mu, x^{\mu+\mu_1}, \cdots, x^{\mu+\mu_{N-1}}; \cdots; x^{m\mu}, x^{m\mu+\mu_1}, \cdots, x^{m\mu+\mu_{N-1}}, \cdots$, where $0 < \mu_1 < \mu_2 < \cdots < \mu_{N-1} < \mu$, a recursion formula and hence a Christoffel-Darboux formula can be given for the orthonormal sequence. It is then possible to construct a theory for point-by-point convergence in the usual way. Since, on the one hand, there are no non-trivial results on boundedness of $p_{\mu_n}(x)$; and, on the other hand, no very interesting extensions of Bernstein's and Markoff's theorems are available, the results which are obtainable are omitted.

[10] D. Jackson, *Certain problems of closest approximation*, this Bulletin, vol. 39 (1933), pp. 903–906, Theorems 11, 15.

[11] It is a well known fact that $\overline{Q}_{\mu_n}(x) = \sum_{r=0}^{n} a_r p_{\mu_r}(x)$. Statements (A) and (B) therefore refer to the error of approximation by partial sums of the series for $f(x)$.

$$\text{(A)} \qquad |f(x) - \bar{Q}_{\mu_n}(x)| \leqq C_1\mu_n\epsilon_n, \qquad a \leqq x \leqq b,$$

$$\text{(B)} \qquad |f(x) - \bar{Q}_{\mu_n}(x)| \leqq C_2\mu_n^{1/2}\epsilon_n, \qquad c \leqq x \leqq d,$$

where $a<c<d<b$.

The problem of convergence then reduces to one of placing sufficient conditions on $f(x)$ so that $\mu_n\epsilon_n \to 0$ or $\mu_n^{1/2}\epsilon_n \to 0$. It is assumed in the following theorems that $\rho(x)$ is bounded from zero and integrable. From the results of the preceding sections values of ϵ_n are obtained which lead to the following theorems.

THEOREM IV. *If $f(x)$ has a derivative for which $x^{1-H}f'(x)$ satisfies a Lipschitz condition H on $[a, b]$, then the series* (1) *converges uniformly to $f(x)$ throughout $[a, b]$, and the magnitude of the remainder after n terms does not exceed $O[1/(\mu_n - H)]$.*

THEOREM V. *If $f(x)$ has a derivative for which $x^{1-H}f'(x)$ is continuous on $[a, b]$ and has a modulus of continuity H, $\omega_H(\delta)$,* (1) *converges uniformly to $f(x)$ throughout $[a, b]$, and the magnitude of the remainder after n terms does not exceed $O[\omega_H((b^H - a^H)/(\mu_n - H))]$.*

THEOREM VI. *If $f(x)$ satisfies a Lipschitz condition H on $[a, b]$,* (1) *converges uniformly to $f(x)$ throughout $[c, d]$, where $a<c<d<b$, and the magnitude of the remainder after n terms does not exceed $O(1/\mu_n^{1/2})$.*

THEOREM VII. *If $f(x)$ has a modulus of continuity H, $\omega_H(\delta)$, for which $\lim_{\delta=0} \omega_H(\delta)/\delta^{1/2} = 0$,* (1) *converges uniformly to $f(x)$ on $[c, d]$, where c and d have the same meaning as before.*

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