it would be quite easy to use various topics treated in the book in a course, whose main interest is not integral operators. One might mention, the Riemann-Stieltjes integral, functions of bounded variation, methods of summation of series, positive definite series, the moment problems, Bernstein's theorem, the Tauberian theorems, the prime number theorem, the Laguerre polynomials, the notion of a positive definite kernel of an integral equation, and the specific integral equations mentioned.

Thus the author has presented us with a treatise on a branch of analysis of great importance and whose applications are of wide interest. The book is extremely satisfactory, when concerned with either its principal topics or the other related developments and one is confident that it will have a most valuable effect both on research and graduate study.

F. J. MURRAY

Mathematical Methods in Engineering. By Theodore von Kármán and Maurice A. Biot. New York, McGraw-Hill, 1940. 12+505 pp. $4.00.

This book by two masters of applied mathematics selects certain representative groups of advanced engineering problems and presents the appropriate mathematical methods of solution, together with helpful excursions into purely mathematical topics. This is a very effective plan that might well be followed by future books in applied mathematics. The problems are largely in mechanics and are interesting, instructive and up-to-date, especially those on the airplane—as might be expected from the interests and accomplishments of the senior author.

Besides the problems worked out in the text, each chapter includes a set of problems of graded difficulty to be worked out by the student, the answers being given at the end of the book. Each chapter opens with a thought-provoking quotation from an authority in the field and ends with a well selected list of references, mostly standard texts.

The authors are as careful with their mathematics as with their physics and engineering, but have chosen to omit mathematical proofs in many cases where they felt that space was not available. It seems to the reviewer that in some of these cases at least a brief outline of the proof could have been given to the advantage of the reader.

Chapter I is a direct, clear introduction to ordinary differential equations. Numerical solutions are properly emphasized; but the step-by-step method given is not the most suitable for ordinary engineering purposes. This method involves the calculation of several initial values by a Taylor series, and so is not applicable to differential equations with empirical coefficients, which are very common in engineering. Furthermore, it is based on an elaborate difference table, which means tedious calculations and requires equal steps in the argument. The latter limitation makes it unsuitable for a function whose derivative varies over a wide range and inapplicable where the derivative becomes infinite. (Actually the example worked out on on p. 19 could have been done with less work by using the Taylor series throughout.) Numerical methods for systems of first-order differential equations (including single higher-order differential equations) are not given, although these are common in engineering.

Simpson's rule is given as the preferred method for numerical quadrature; and an example on p. 6 shows that it gives 5-digit accuracy as compared with 2-digit accuracy for the simple trapezoidal rule. However, the inaccuracy of the trapezoidal rule is mainly due to lack of end corrections: if the single correction term $(1/12)h^2f'(x)\big|_0^1$ is included, the trapezoidal rule here gives 7-digit accuracy. (This correction term is the first of a series in the Euler summation formula; it can also be derived geometrically from the assumption that the integrand $f(x)$ is sufficiently approximated by a parabolic arc.) In the reviewer's opinion, Simpson's rule should be abandoned as not giving the best approximation obtainable from a given set of values.

On p. 7, it is stated that by means of isoclines a step-by-step solution of $D_xy=f(x, y)$ can always be obtained "provided that $f(x, y)$ is a single-valued continuous function of x and y." The implication that a unique solution can be obtained is incorrect because the Lipschitz condition has been disregarded. For example, the right-hand member of the equation $D_xy=3y^{1/2}-x$, with $y^{1/2}$ restricted to positive values, is single-valued and continuous at (0, 0) but there is an infinity of solutions starting at this point, given by $y^{1/2}=x+C-(Cx+C^2)^{1/2}$.

Chapter II treats of Bessel functions as the solutions of Bessel's equation. The power series are derived by the method of undetermined coefficients, without a test for convergence nor a proof that the series actually satisfies the differential equation, although the method of successive substitution (or its operational equivalent) would both be shorter and would provide a remainder permitting rigorous treatment. Contour integrals are not employed; so no proof is given for the relations between the coefficients of the converging and the asymptotic series. However, the description of the properties of the Bessel functions is good and is illustrated by helpful graphs. Also there are given various common equivalent forms of Bessel's equations, with their solutions. The notation used is that of Watson.

Chapter III is devoted to the fundamental concepts of dynamics and includes Lagrange's equations in generalized coordinates. It starts with Newton's laws of motion and later derives the principle of conservation of energy for a mechanically conservative system. This is the classical approach; but, inasmuch as conservation of energy is the most basic principle of all physics, it would seem sounder philosophically to start with it as an experimental fact (as the reviewer has done for some years in his physics classes). Newton's third law of motion is, in fact, essentially a statement of conservation of energy as applied to mechanical systems—that when a body A exerts a force on a body B, the energy taken out of A in a joint displacement of the two bodies is equal to the energy put into B. (As has been recognized by Kelvin and Tait, Newton, in the Scholium following the enunciation of his laws, implicitly anticipated the concepts of energy and power.)

Newton's second law is given in the form $F=D_tmv$ and reference is made to the fact that m as well as v may be variable. This generalization, however, is treacherous: it applies, for example, to a falling raindrop whose mass is increasing by accretion from water vapor; but it does not apply to a leaking water bucket, where the correct form is $F=mD_tv$, even though m is varying.

Theorems are worked out for mass points (particles); and it is then stated without proof (p. 93) that they also apply to continuous fluid and elastic bodies. This is taking a big jump. Had energy been used from the start, all theorems could have been proved rigorously.

Much of the development in this chapter is in vector notation, including scalar $(\cdot)$ and vector $(\times)$ products.

Chapter IV discusses various problems in dynamics and includes two purely mathematical discussions: (1) elliptic integrals, introduced to express the motion of a pendulum; and (2) the singularities of first-order differential equations. Both of these discussions are clear and interesting and they gain by their association with important physical problems.

Chapter V is on the small oscillations of conservative systems and employs the Lagrange equations. In connection with the numerical calculation of natural frequencies, the following methods of finding the real roots of algebraic equations are given: Newton's, iteration, Graeffe's. Matrices are also introduced to find the frequencies and the normal modes by successive approximations. The only matrix manipulation used is the product of a square matrix by a column matrix; and no proof is given for the rule.

In Chapter VI, small oscillations of non-conservative systems are taken up; and dynamic stability is treated, calling for the calculation of the complex roots of algebraic equations with real coefficients.

Chapter VII discusses the differential equations of various continuous structures, such as strings and beams, with and without uniformly distributed elastic restraints. Harmonic vibration is shown to lead to the same forms of equation as these elastic restraints. Other problems considered are the buckling of uniform and tapered columns and the combined axial and lateral loading on the spar of an airplane wing.

In Chapter VIII, the Fourier series and integral are applied to continuous structures. It is shown that the Fourier coefficients give the best approximation to an arbitrary function, for a terminated trigonometric series, and that the mean-square error steadily decreases as terms are added. Sufficient conditions for convergence are stated without proof; and the Gibbs phenomenon is described. Numerical, graphical and mechanical methods for determining the coefficients are described. This chapter also discusses the Rayleigh-Ritz method of approximating natural frequencies and characteristic functions, applying it to trigonometric series.

Chapter IX applies complex numbers to periodic phenomena, both mechanical and electrical. The concept of impedance is properly emphasized, but unfortunately two kinds of mechanical impedance are used, force/displacement and force/velocity; and the former is preferred, although it is not the analogue of electrical impedance. This confusion is quite unnecessary, as the two impedances are related simply by the factor $i\omega$. For arbitrary impressed forces or voltages, the Fourier series is introduced in complex form.

Chapter X is devoted to the Heaviside operational calculus, which is applicable to linear differential equations (ordinary or partial) with constant coefficients. Various procedures have been used for establishing the methods of this calculus: (a) the direct operational approach; (b) Fourier integrals and transforms (or the equivalent Laplace transform); (c) the Bromwich contour integral; and (d) Carson's integral equation. Of these, procedure (a) is the simplest for ordinary differential equations, which is as far as this chapter goes: it involves only the separation of rational functions of the derivative operator D into partial fractions and the use of the single evaluating formula $(D-\alpha)^{-q-1}DH(t)=(t^q/q!)\epsilon^{\alpha t}H(t)$, where $H(t)$ is the Heaviside function or unit-step function. If the operand consists of terms of the form $t^q\epsilon^{\alpha t}H(t)$, it can be put back into operational form by using this formula backwards, after which partial fractions are applied to the resulting combined operator. If the operand $f(t)$ has another form, any rational operation $g(D)f(t)$ is evaluated as a single integral by the superposition formula $g(D)f(t)=\int_{-\infty}^{\infty}f(\tau)g(D)DH(t-\tau)d\tau$. The proofs of these formulas and of the legitimacy of the algebraic manipulation of the operators are simple by the direct operational approach, making use of the principle characteristic of the Heaviside operational calculus, that the operands $f(t)$ are zero for all sufficiently negative values of t (commonly for all $t<0$). This is a natural condition in many engineering problems and can be introduced artificially in other cases, for example where the physical operand is not defined for $t<0$.

The book under review unfortunately uses more difficult procedures, first employing the Fourier integral and leading up to the Bromwich integral and then proving the operational formulas by Carson's integral equation. The symbol p is confusingly used both as a complex variable and for the derivative operator D. [It is a common misconception that there is a distinction between inverse functions of Heaviside's p and of D, which arose from a failure to restrict operands as in the preceding paragraph. With such restriction and with D^{-1} defined as $\int_{-\infty}^{t}$, all functions of D become perfectly definite and commutative with one another; and no reason remains for a special symbol in place of the ordinary D.] The awkward italic $l(t)$ is used for the Heaviside function $H(t)$; and $S(t)$ is used for the unit impulse function, which is better represented as $DH(t)$, since $D^{-1}S(t)=H(t)$. These methods and usages, of course, are found commonly in the literature; but they seem to be delaying the incorporation of the very valuable Heaviside calculus into the current stream of mathematics.

A too little appreciated branch of applied mathematics is introduced in Chapter XI, the calculus of finite differences. However, the method used is that of assuming the form of a solution and finding the constants which satisfy the given difference equation and the given conditions. This is a field to which operational methods are particularly applicable, especially in a form analogous to the Heaviside operational calculus for differential equations, the evaluating formula being $(1-\alpha E^{-1})^{-q-1}(1-E^{-1})H(t)=[(t+1)\cdots(t+q)/q!]\alpha^{+}H(t)$, now where E is Boole's operator defined by $E^{-1}f(t)=f(t)-1$ and $H(t)$ equals 0 for negative integer t and equals 1 for zero and positive integer t. The problems include deflections of continuous beams supported at equal intervals, the buckling of a rectangular lattice truss, voltage distribution in a suspension insulator, critical speeds of a multicylinder engine, leading to a mechanical wave filter, and electric wave filters. (In the last, an attenuator is incorrectly called a "wave trap," which is a resonant element.)

Following the chapters appears a short section entitled "words and phrases" intended to give certain "strictly mathematical definitions" and starting with Kronecker's dictum, "God made the integers; all the rest is the work of man." It seems surprising to have this dictum endorsed by engineers and physicists: it might better be replaced by the statement, *God made both discrete and continuous physical quantities; man devised means (integers and real numbers) for representing them.* This section in general seems out of place and of little help to a reader who would be capable of using the book.

Besides a few obvious typographical errors, the reviewer noticed the following errors: last equation, p. 108, a minus sign should precede the first term in each member; the answer to Problem 12, p. 108, has the inequality reversed; the answer to Problem 16, p. 109, not only uses k for K but incorrectly gives a stable solution; the first answer to Problem 1, p. 210, incorrectly has the factor 1/2; the equation on p. 131 is written as if t were under the radical sign. (It is unfortunate that radical signs are used in place of fractional exponents throughout the book.)

This is a book which should be in the library of every engineer who is interested in the analytical development of his subject. It should be studied by mathematicians who are willing to admit that their place in society may need justification on other than purely intellectual grounds. It is well adapted as a text or for collateral study in an advanced course in applied mathematics or in theoretical mechanics. The authors are to be congratulated in so competently supplying a real need.

ALAN HAZELTINE

Development of the Minkowski Geometry of Numbers. By Harris Hancock. (Published with the aid of the Charles Phelps Taft Memorial Fund and of two Friends.) New York, Macmillan, 1939. 24+839 pp. $12.00.

Professor Hancock says in the introduction of his book: "In every subject that occupies the human mind, be it history, philosophy law, medicine, science, music, etc., there arise outstanding men who evince an innate genius in their special fields, an innateness that seems as it were of divine origin. Minkowski was one of the great mathematicians of all time." It is the aim of Hancock's book to make an