is of course the multiplication in A_1). By the proof of Theorem 2, in order to show the equivalence of A and A_1 it is sufficient to show that $[w, w]=\gamma f$ and $[w, z]=[zU, w]$ for every z of R. But $[w, w]=w(f^{-1}w)=(fg)(f^{-1}fg)=fg^2=\gamma f$, and $[w, z]=w(f^{-1}z)=(fg)(f^{-1}fx)=(fg)x=g(x\cdot fS)=(f\cdot xS)g=(f\cdot xS)(f^{-1}fg)=zU(f^{-1}w)=[zU, w]$. This proves the theorem.

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ON FIBRE SPACES. I

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In subsequent papers I propose to investigate various properties of fibre spaces.[1] The object of the fundamental Hurewicz-Steenrod definition[1] is to state a minimum[2] set of readily verifiable conditions under which the covering homotopy theorem[1] holds. An apparent defect of their definition is that it is not topologically invariant. In fact, for topological space X and metrizable non-compact space B the property "X is a fibre space over B" depends on the metric of B. The object of this note is to give a topologically invariant definition of fibre space and to show that (when B is metrizable) X is a fibre space over B in this sense if and only if B has a metric in which X is a fibre space over B in the sense of Hurewicz-Steenrod. Since the definition of fibre space is controlled by the covering homotopy theorem, an essential part of my program is to give a topologically invariant definition of uniform homotopy.

Let π be a continuous mapping of a topological space X into another topological space B. Let $\Delta=\Delta(B)$ denote the diagonal set $\sum_{b\in B}(b, b)$ of the product space $B\times B$ and let $\bar{\pi}$ denote the mapping of $X\times B$ into $B\times B$ which is induced by the mapping π according to the rule $\bar{\pi}(x, b)=(\pi(x), b)$. Thus the graph G of π is the set $\bar{\pi}^{-1}(\Delta)$, and $\bar{\pi}^{-1}(U)$ is a neighborhood of G whenever U is a neighborhood of Δ.

Any neighborhood U of Δ determines uniquely a covering of B by neighborhoods $N_U(b)$ according to the rule $b'\in N_U(b)$ when $(b, b')\in U$.

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[1] W. Hurewicz and N. E. Steenrod, Proc. Nat. Acad. Sci. U.S.A. vol. 27 (1941) p. 61.

[2] How well they succeeded in this will be indicated in my next communication.

However not every covering of B by neighborhoods need arise in this fashion—although the star neighborhoods of any open covering of B may always be so generated.

A *slicing function* ϕ for π is any continuous mapping defined over $\bar{\pi}^{-1}(U)$ for some neighborhood U of Δ, with values in X, which satisfies the conditions

$$\pi\phi(x, b) = b,$$
$$\phi(x, \pi(x)) = x,$$

whenever ϕ is defined. I shall call π a *fibre mapping* relative to U if it has a slicing function defined over $\bar{\pi}^{-1}(U)$. If π is a fibre mapping I shall say that X is a *fibre space* over the subset $\pi(X)$ of B. Since U is a neighborhood of Δ, $\pi(X)$ is open and closed in B.

This new definition is equivalent to the old one if the base space is compact metric (so that the Hurewicz-Steenrod definition is topologically invariant in this case). In fact, for metric space B, let σ_ϵ denote that neighborhood of Δ which determines the covering of B by ϵ-spheres. Clearly X is a fibre space (relative to π) over the metric space $\pi(X)$ in the sense of Hurewicz-Steenrod if and only if π has a slicing function defined over $\bar{\pi}^{-1}(\sigma_\epsilon)$ for some $\epsilon > 0$. Hence, *if π is a fibre mapping and $\pi(X)$ is compact metrizable then X is a fibre space over $\pi(X)$ in the sense of Hurewicz-Steenrod no matter how $\pi(X)$ is metrized.*

Now let B denote an arbitrary metrizable space, let U be a neighborhood of Δ and let π be a fibre mapping whose slicing function is defined over $\bar{\pi}^{-1}(U)$. For simplicity, assume also that $\pi(X) = B$. To show that X is a fibre space in the sense of Hurewicz-Steenrod when B is properly metrized it is clearly sufficient to so metrize B that $\sigma_\epsilon \subset U$ for some $\epsilon > 0$.

LEMMA.[3] *If B is metrizable and U is an open neighborhood of $\Delta(B)$ then B can be so metrized that $\sigma_1 \subset U$.*

Choose any random metric d for B. Since $B \times B$ is metric, hence normal, it is possible to define a continuous function $f \in [0, 1]^{B\times B}$ such that

$$f(b, b_0) = \begin{cases} 0 & \text{when } (b, b_0) \in \Delta, \\ 1 & \text{when } (b, b_0) \in B \times B - U. \end{cases}$$

Let ϕ denote the (continuous) mapping $b \to f_b$, where $f_b(b_0) = f(b, b_0)$.

[3] This proof is modelled after a proof in André Weil, *Sur les espaces à structure uniforme et sur la topologie générale*, Actualités Scientifiques et Industrielles, no. 551, 1938, p. 15.

The graph $B'=\sum_{b\in B}(b, \phi(b))$ of ϕ is homeomorphic to B. The metric of B' is induced by the metric of the product $B\times[0, 1]^B$ and is given by the formula

$$\delta(b_1', b_2') = \{d^2(b_1, b_2) + d^2(\phi(b_1), \phi(b_2))\}^{1/2},$$

where b and b' denote corresponding points of B and B'. If now $(b_1', b_2')\in\sigma_1$ then $\delta(b_1', b_2')<1$, hence $d(\phi(b_1), \phi(b_2))<1$, hence $\sup_{b\in B}|f(b_1, b)-f(b_2, b)|<1$. It follows that $f(b_1, b_2)=|f(b_1, b_2)-f(b_2, b_2)|<1$, so that $(b_1, b_2)\in U$ and $(b_1', b_2')\in U'$.

THEOREM. *If π is a fibre mapping and B is metrizable then the metric of B can be so chosen that X is a fibre space over $\pi(X)$ (relative to π) in the sense of Hurewicz and Steenrod.*

I conclude by defining uniform homotopy and stating the covering homotopy theorem for general fibre spaces. If h is a homotopy in B of a space Y and U is a neighborhood of Δ I shall say that h is *uniform* with respect to U if there is a $\delta>0$ such that $|t-t'|<\delta$ implies that $\sum_{y\in Y}(h(y, t), h(y, t'))\subset U$. Let $E_\delta=\sum_{0\leqq t,t'\leqq 1,|t-t'|<\delta}\sum_{y\in Y}(h(y, t), h(y, t'))$, so that $E_0\subset\Delta$ and $E_1\subset B\times B$. Clearly the neighborhoods U with respect to which h is uniform are those which contain an E_δ for some $\delta>0$. Thus h is always uniform with respect to $B\times B$; in the event that Y is compact h is uniform with respect to every neighborhood U. I shall call a homotopy h^* in X a *covering homotopy* (with respect to π) if

(1) $\pi h^*=h$,

(2) $h^*_{[0, 1]}(y)$ degenerates to a point whenever $h_{[0,1]}(y)$ degenerates to a point.

I shall refer to the mappings h_0 and h_0^* as the initial values of the homotopies h and h^*, respectively. With these notations the covering homotopy theorem for fibre mappings reads thus.

THEOREM. *Given a fibre mapping $\pi\in B^X$ relative to U, a mapping $g\in X^Y$ and a homotopy h in B, uniform with respect to U, with initial value πg, there exists a covering homotopy h^* in X with initial value g.*

The covering homotopy h^* is constructed stepwise[1] and is easily seen to be uniform with respect to $U^*=\ddot{\pi}^{-1}(U)$ where $\ddot{\pi}(x, x')=(\pi(x), \pi(x'))$. Of course if U is a σ_ϵ the neighborhood U^* of $\Delta(X)$ need not be a $\sigma_\epsilon(X)$.

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