CAUCHY-STIELTJES AND RIEMANN-STIELTJES INTEGRALS

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1. Introduction. This note treats the equivalence of the Riemann-Stieltjes and Cauchy-Stieltjes integrals (abbreviated RS and CS integrals) and conditions for the existence and equality of the latter. The ordinary Riemann, the left Cauchy, and the right Cauchy integrals are defined as limits of the sums $\sum_1^n f(\xi_i)(x_i - x_{i-1})$, $x_{i-1} \leqq \xi_i \leqq x_i$, $\sum_1^n f(x_{i-1})(x_i - x_{i-1})$ and $\sum_1^n f(x_i)(x_i - x_{i-1})$ respectively; it is known [2] that these integrals are equivalent. Corresponding to these integrals, we have the RS and the two CS integrals, defined as limits of the sums $\sum_1^n f(\xi_i)[g(x_i) - g(x_{i-1})]$, $\sum_1^n f(x_{i-1})[g(x_i) - g(x_{i-1})]$, and $\sum_1^n f(x_i)[g(x_i) - g(x_{i-1})]$. The right modified RS integral is obtained from the sums $\sum_1^n f(\xi_i)[g(x_i) - g(x_{i-1})]$, $x_{i-1} \leqq \xi_i < x_i$. Examples in §4 show that the CS integrals may exist, with equal or unequal values, when the RS and the right modified RS do not; that the right modified RS integral may exist when the RS does not; and that one of the CS integrals may exist when the other does not. Thus the RS, the right modified RS, and the two CS integrals are not equivalent. Since all these integrals obviously exist when the RS integral does, it is natural to investigate conditions under which the existence of a CS or right modified RS integral implies the existence of the RS integral. It is shown in this note that if g is non-decreasing, if f and g have no common discontinuities on the same side, and if the left CS integral exists, then the RS integral exists and has the same value, the integrals being limits in the sense of increasing refinement of subdivisions. This result is established in two steps: (a) if g is non-decreasing, if f and g have no common discontinuities on the right, and if the left CS integral exists, then the right modified RS integral exists; (b) if the right modified RS integral exists, if g is non-decreasing, and if f and g have no common discontinuities on the left, then the RS integral exists. This result obviously includes the previously proved equivalence of the Riemann and Cauchy integrals [2] and certain others [3]. Further, it states sufficient conditions for the equality of the two CS integrals; these conditions show that the two ordinary Cauchy integrals are always equal. The note closes with a proof that the CS integrals exist when f has only simple discontinuities and g has bounded variation. We conclude from this result and others stated above that both of the CS integrals properly include the RS integral, and that neither CS integral includes the other. Precise statements of

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theorems are given in §5.

2. **Notation.** Let f and g denote real, single valued, bounded functions defined on $a \leqq x \leqq b$. Let X denote a subdivision $a = x_0 < x_1 < \cdots < x_n = b$ of the interval $a \leqq x \leqq b$, and let $x' \leqq x \leqq x''$ denote a generic interval of X. Let $X_2 \geqq X_1$ denote that X_2 is a refinement of X_1, and let X_1X_2 denote the subdivision determined by all the points of division of X_1 and X_2. Also let $\|X\|$ denote the norm of X, that is, the maximum length of a subinterval of X. Finally, let $X\Psi(\Delta)$ denote the sum of the values of the interval function Ψ for all the intervals Δ of X. The interval functions to be met below are of the form $f(\xi)[g(x'')-g(x')]$, $x' \leqq \xi \leqq x''$. This notation has been used by Geöcze [1].

3. **Definitions.** In this section we give definitions of various integrals.

Definition. *We say f has a left* CS X *integral with respect to g, denoted by $(C_L, X)\int_a^b fdg$, if and only if for each $\epsilon > 0$ there exists an $X_0(\epsilon)$ such that*

$$(1) \qquad \left| (C_L, X)\int_a^b fdg - Xf(x')[g(x'') - g(x')] \right| < \epsilon, \qquad X \geqq X_0(\epsilon).$$

Definition. *We say f has a left* CS $\|X\|$ *integral with respect to g, denoted by $(C_L, \|X\|)\int_a^b fdg$, if and only if for each $\epsilon > 0$ there exists a $\delta(\epsilon) > 0$ such that*

$$(2) \qquad \left| (C_L, \|X\|)\int_a^b fdg - Xf(x')[g(x'') - g(x')] \right| < \epsilon, \qquad \|X\| \leqq \delta(\epsilon).$$

The definitions of the corresponding right CS integrals $(C_R, X)\int_a^b fdg$, $(C_R, \|X\|)\int_a^b fdg$ are obtained by replacing $f(x')$ by $f(x'')$ in (1) and (2). Also, if $f(x')$ is replaced by $f(\xi)$, $x' \leqq \xi \leqq x''$, we obtain the RS integrals $(R, X)\int_a^b fdg$, $(R, \|X\|)\int_a^b fdg$. If $f(x')$ in (1) is replaced by $f(\xi)$, $x' \leqq \xi < x''$, we have the right modified RS X integral $(R^*, X)\int_a^b fdg$.

4. **Examples.** A first example shows that the CS integrals may exist, with equal or unequal values, even when the RS and the right modified RS integrals do not, and that the right modified RS integral may exist when the RS does not. Let f have the value 1 when $x = 0$ and the value 0 elsewhere on the interval $-1 \leqq x \leqq 1$. Set $g(0) = c$, $g(x) = -1$ on $-1 \leqq x < 0$, and $g(x) = 1$ on $0 < x \leqq 1$. Then $(C_L, X)\int_{-1}^1 fdg = 1 - c$, $(C_R, X)\int_{-1}^1 fdg = 1 + c$. The $\|X\|$ CS integrals do not exist, and neither do the X and $\|X\|$ RS integrals. If $c = 1$, then $(R^*, X)\int_{-1}^1 fdg$

exists and equals $(C_L, X)\int_{-1}^{1} fdg$.

A second example shows that one CS integral may exist when the other does not. Let $f(x)=\sin(1/x)$ and $g(x)=x-1$ on $-1\leqq x<0$; $f(x)=0$ and $g(x)=x+1$ on $0\leqq x\leqq 1$. Then $(C_R, X)\int_{-1}^{1} fdg$ exists but $(C_L, X)\int_{-1}^{1} fdg$ does not (see a necessary condition for the existence of the latter [**3**, p. 267, 2.14]).

5. **Theorems.** The results to be established will be stated in this section.

THEOREM 1. *Let f and g, defined on $a\leqq x\leqq b$, have the following properties:*

(i) $|f(x)|\leqq M$;

(ii) *$g(x)$ is non-decreasing;*

(iii) *f and g have no common discontinuities on the left;*

(iv) *$(R^*, X)\int_a^b fdg$ exists.*

Then $(R, X)\int_a^b fdg$ exists and equals $(R^*, X)\int_a^b fdg$.

THEOREM 2. *Let f and g satisfy* (i), (ii) *and the following:*

(v) *f and g have no common discontinuities on the right;*

(vi) *$(C_L, X)\int_a^b fdg$ exists.*

Then $(R^*, X)\int_a^b fdg$ exists and equals $(C_L, X)\int_a^b fdg$.

COROLLARY 1. *Let f and g satisfy* (i), (ii), (vi), *and the following:*

(vii) *f and g have no common discontinuities on the same side.*

Then $(R, X)\int_a^b fdg$, $(C_R, X)\int_a^b fdg$ exist and equal $(C_L, X)\int_a^b fdg$.

COROLLARY 2. *If the hypotheses of Corollary* 1 *are strengthened by replacing* (vii) *by*

(viii) *f and g have no common discontinuities,*

then $(R, \|X\|)\int_a^b fdg$ also exists.

THEOREM 3. *If f satisfies* (i) *and has only simple discontinuities, and if g has bounded variation, then $(C_L, X)\int_a^b fdg$ and $(C_R, X)\int_a^b fdg$ exist.*

6. **Proofs of the theorems.** The proof of Theorem 1 will be omitted; a more general result is known [**3**, p. 273].

Consider Theorem 2. Let ϵ be an arbitrary positive constant, and let $X_0(\epsilon)$ be the subdivision which exists by (1) and (vi). Let η be another arbitrary positive number. In each interval $x'\leqq x\leqq x''$ of X_0 define a point t as follows: (a) $t=x''$ if $f(x)\geqq f(x')+\eta$ has no solution x satisfying $x'<x<x''$; (b) $t=\text{g.l.b. } E_x[f(x)\geqq f(x')+\eta, x'<x<x'']$ in case this set is not empty. With each point t associate a sequence of points $t_1, t_2, \cdots$ as follows: (a) $t_k=t$, $k=1, 2, \cdots$, if $t=x''$; (b) $t_k=t$, $k=1, 2, \cdots$, if $f(t)\geqq f(x')+\eta$; (c) in the remaining case

$\{t_k\}$ is a sequence such that $f(t_k) \geqq f(x') + \eta$, $k = 1, 2, \cdots$, and $\lim_k t_k = t$. The third case occurs only when f is discontinuous on the right at t; at such a point g is continuous on the right by (v). Then for every interval of X_0 $\lim_k [g(x'') - g(t_k)] = [g(x'') - g(t)]$. This fact will be used to prove

$$X_0 \eta [g(x'') - g(t)] \leqq 2\epsilon. \tag{3}$$

Let X_1 be the subdivision obtained by adding the points of division t to those of X_0, and let X_1^k be the subdivision obtained from X_0 by subdividing each of its intervals by t_k. Since $X_1^k \geqq X_0$, we have

$$| X_1^k f(x') [g(x'') - g(x')] - X_0 f(x') [g(x'') - g(x')] | < 2\epsilon \tag{4}$$

by (1) and (vi). But since

$$X_1^k f(x') [g(x'') - g(x')] = X_0 f(x') [g(t_k) - g(x')] + X_0 f(t_k) [g(x'') - g(t_k)],$$

we find that the left-hand side of (4) reduces to

$$| X_0 [f(t_k) - f(x')] [g(x'') - g(t_k)] | \geqq X_0 \eta [g(x'') - g(t_k)]. \tag{5}$$

Thus from (4) and (5) we find $X_0 \eta [g(x'') - g(t_k)] < 2\epsilon$; by taking the limit as $k \to \infty$, we obtain (3).

Let $U^*(f; x', x'')$ and $L^*(f; x', x'')$ denote the l.u.b. and g.l.b. respectively of f on $x' \leqq x < x''$. Then since

$$\begin{aligned} X_1 U^*(f; x', x'') [g(x'') - g(x')] \\ = X_0 U^*(f; x', t) [g(t) - g(x')] + X_0 U^*(f; t, x'') [g(x'') - g(t)], \end{aligned}$$

we have

$$\begin{aligned} | X_1 U^*(f; x', x'') [g(x'') - g(x')] - X_0 f(x') [g(x'') - g(x')] | \\ \leqq | X_0 [U^*(f; x', t) - f(x')] [g(t) - g(x')] | \\ + | X_0 [U^*(f; t, x'') - f(x')] [g(x'') - g(t)] | \\ \leqq \eta [g(b) - g(a)] + 2M X_0 [g(x'') - g(t)] \leqq \eta [g(b) - g(a)] + 4M\epsilon/\eta \end{aligned}$$

by (i), (ii), (3), and the definition of t. Thus

$$\begin{aligned} \left| X_1 U^*(f; x', x'') [g(x'') - g(x')] - (C_L, X) \int_a^b f dg \right| \\ \leqq \eta [g(b) - g(a)] + 4M\epsilon/\eta + \epsilon \end{aligned}$$

by the result just established and the definition of X_0. If ζ is an arbitrary positive number, it is possible to choose η and ϵ so small that the expression on the right of the last inequality is less than ζ. By a similar argument we can show that there exists a subdivision X_2 such that

$$\left| X_2 L^*(f;\, x',\, x'')[g(x'') - g(x')] - (C_L, X)\int_a^b fdg \right| < \zeta.$$

It follows from these results that

$$\left| Xf(\xi)[g(x'') - g(x')] - (C_L, X)\int_a^b fdg \right| < \zeta$$

for $x' \leqq \xi < x''$ and any $X \geqq X_1 X_2$. Thus $(R^*, X)\int_a^b fdg$ exists and equals $(C_L, X)\int_a^b fdg$, and the proof of Theorem 2 is complete.

Corollary 1 follows from Theorems 1 and 2. A sufficient condition that $(R, \|X\|)\int_a^b fdg$ exist is that $(R, X)\int_a^b fdg$ exist, and that f and g have no common discontinuities [**3**, p. 269, 4.14]; Corollary 2 follows from this condition and Corollary 1.

Before we begin the proof of Theorem 3, two preliminary results are needed. Let g be any function of bounded variation on $a \leqq x \leqq b$, and let x' be any point such that $a \leqq x' < b$. Then given any $\zeta > 0$, there exists an x'' sufficiently near x' on the right so that the total variation of g on any interval $t \leqq x \leqq x''$, where $x' < t < x''$, is less than ζ. To prove this statement we define $h(x) = g(x)$, $x \neq x'$, and $h(x') = g(x'+0)$ and recall that the total variation of $h(x)$ is continuous on the right at x' [**6**, p. 356]. The second preliminary result is the following. If f is defined on $\alpha \leqq x \leqq \beta$ and has simple discontinuities only, and if the oscillation of f at each point of the interval $\alpha \leqq x < \beta$ is less than λ, then $\alpha \leqq x \leqq \beta$ can be subdivided into a finite number of intervals $x' \leqq x \leqq x''$ such that the oscillation of f on $x' \leqq x < x''$ is less than 2λ. The proof follows readily from a theorem of Baire [**4**, p. 311].

Finally, consider the proof of Theorem 3. Let any $\epsilon > 0$ be given. Form a subdivision $X_0(\epsilon)$ of $a \leqq x \leqq b$ by points $a = x_0 < x_1 < \cdots < x_n = b$ so that (a) each of the r points at which the oscillation of f is equal to or greater than λ is a point of division; (b) if $x' \leqq x \leqq x''$ is an interval of X_0 such that the oscillation of f at x' is equal to or greater than λ, the total variation of g on any interval $t \leqq x \leqq x''$, $x' < t < x''$, is less than ζ; (c) if $x' \leqq x \leqq x''$ is not an interval of the type described in (b), the oscillation of f on $x' \leqq x < x''$ is less than 2λ.

Let X_1 be any refinement of X_0. A straightforward calculation shows that

$$(6)\quad \left| X_1 f(x')[g(x'') - g(x')] - X_0 f(x')[g(x'') - g(x')] \right| < 2Mr\zeta + 2\lambda V,$$

where V is the total variation of g on $a \leqq x \leqq b$. We may suppose that

λ was chosen so that $2\lambda V<\epsilon/4$. This choice of λ determines the integer r. We may suppose further that ζ was chosen so that $2Mr\zeta<\epsilon/4$. Then the right-hand member of (6) is less than $\epsilon/2$. Similarly, for $X_2 \geqq X_0$, we have

$$\left| X_0 f(x')[g(x'') - g(x')] - X_2 f(x')[g(x'') - g(x')]\right| < \epsilon/2;$$

hence

$$\left| X_1 f(x')[g(x'') - g(x')] - X_2 f(x')[g(x'') - g(x')]\right| < \epsilon, \qquad X_1, X_2 \geqq X_0(\epsilon).$$

This is a sufficient condition that $(C_L, X)\int_a^b f dg$ exists [5, p. 106]. A similar proof shows that $(C_R, X)\int_a^b f dg$ also exists.

All proofs are now complete.

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