THE COMPACTNESS OF THE RIEMANN MANIFOLD OF AN ABSTRACT FIELD OF ALGEBRAIC FUNCTIONS

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1. The existence of finite resolving systems. In an earlier paper[1] we have announced the result that the existence of a resolving system of the Riemann manifold of an abstract field of algebraic functions (in any number of variables) or—what is the same—the local uniformization theorem[2] implies the existence of *finite* resolving systems of the Riemann manifold. We have proved this result for algebraic surfaces by arithmetic considerations.[1] The proof for the general case of varieties, which at that time was in our possession,[3] and which we have promised to publish in a subsequent paper, was of similar nature, that is, it was based upon considerations involving the structure of certain infinite sequences of quotient rings. However, we have succeeded lately in finding a much simpler proof which is based on topological considerations.

Let Σ be a field of algebraic functions of several variables, over an arbitrary ground field k. By the Riemann manifold M of Σ we mean the totality of places of Σ, that is, the totality of zero-dimensional valuations v of Σ/k. If V is a projective model of Σ/k, and if H is any subset of V, we denote by $N(H)$ the subset of M consisting of those valuations $\mathfrak{v}$ which have center in H. By a resolving system of M we mean a collection $\mathfrak{V} = \{V_\alpha\}$ of projective models (finite or infinite in number) with the property that for any v in M there exists a V_α in $\mathfrak{V}$ such that the center of v on V_α is a simple point.

The topology which we introduce in M is simply this: *we choose as a basis for the closed sets of M the sets $N(W)$, where W is any algebraic subvariety of any projective model of Σ.* We prove that if topologized in this fashion, *the set M is a compact*[4] *topological space.* From this the result announced above follows immediately. For if $\{V_\alpha\}$ is a resolving system, and if we denote by S_α the singular locus of V_α, then $N(V_\alpha - S_\alpha)$ is an open set and $\{N(V_\alpha - S_\alpha)\}$ is an open covering of M.

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[1] *A simplified proof for the resolution of singularities of an algebraic surface*, Ann. of Math. vol. 43 (1942) p. 583.

[2] See loc. cit. footnote 1.

[3] That proof was presented by us at a seminar in algebraic geometry at Johns Hopkins in 1942.

[4] We use the term compact in the same sense as it is used by S. Lefschetz in his *Algebraic topology* (Amer. Math. Soc. Colloquium Publications, vol. 27, 1942). The old term is bicompact.

Hence this covering contains a finite subcovering $\{N(V_i - S_i)\}$, $i=1, 2, \cdots, m$, and this means that $\{V_1, V_2, \cdots, V_m\}$ is a finite resolving system of M.

The proof of compactness of M given in the next section is based in part on some simple algebro-geometric considerations, and in part on a theorem of Steenrod[5] on the compactness of the limit space of an inverse system of compact T_1-spaces.

2. **The Riemann manifold as the limit space of an inverse system.** Let $\mathfrak{V} = \{V_a\}$ be the collection of *all* projective models of Σ/k. By a point of V_a we mean a zero-dimensional prime ideal in a suitable coördinate ring of V_a, or, in other terms, a point is a prime one-dimensional *homogeneous* ideal in the ring of homogeneous coördinates of the general point of V_a. This defines V_a set-theoretically as a set of points. We topologize V_a by choosing as closed sets the algebraic subarieties of V_a. It is obvious that V_a then becomes a compact topological space in which points are closed sets (whence V_a is a T_1-space; however, it is not a Hausdorff space).

If V_a and V_b are two projective models of Σ/k, we denote by π_a^b the transformation of V_b onto V_a defined by the birational correspondence between V_a and V_b. We define a partial ordering $\prec$ of the collection $\mathfrak{V}$ as follows: $V_a \prec V_b$ if whenever P_a and P_b are corresponding points of V_a and V_b under π_a^b then $Q(P_a) \subseteq Q(P_b)$. Here $Q(P)$ denotes the quotient ring of P. It is clear that if $V_a \prec V_b$ then π_a^b is a single-valued continuous and *closed* mapping. Moreover, if V_a and V_b are arbitrary projective models of Σ/k, and if V_c denotes the join[6] of V_a and V_b, then $V_a \prec V_c$ and $V_b \prec V_c$. Hence we have here an inverse system $\{V_a; \pi_a^b | V_a \prec V_b\}$ of compact T_1-spaces. Let M be the limit space of the system. By Steenrod's theorem M is compact. Every point P^* of M represents an infinite collection of points $\{P_a\}$, $P_a \in V_a$, $V_a \in \mathfrak{V}$, with the property that if $V_a \prec V_b$ then $Q(P_a) \subseteq Q(P_b)$. We shall denote by π_a^* the mapping $P^* \rightarrow P_a$ of M into V_a. If V_a is any projective model of Σ/k and if W is any algebraic subvariety of V_a, then $(\pi_a^*)^{-1}W$ is a closed subset of M, and the closed sets obtained in this fashion form a basis for the closed subsets of M.

The compactness of the Riemann manifold of Σ/k and the implications stated in the preceding section are immediate consequences of the following theorem.

THEOREM. *There is a* (1, 1) *correspondence between the points* P^*

[5] N. E. Steenrod, *Universal homology groups*, Amer. J. Math. vol. 58 (1936) p. 666.

[6] See our paper *Foundations of a general theory of birational correspondences*, Trans. Amer. Math. Soc. vol. 53 (1943) p. 516.

of M and the zero-dimensional valuations v of the field Σ/k. If P^* and v are corresponding elements, and if V_a is any projective model of Σ/k, then $\pi_a^* P^*$ is the center of v on V_a.

PROOF. Let v be a zero-dimensional valuation of Σ/k and let $P_{a,v}$ be the center of v on any given projective model V_a of Σ/k. For any two projective models V_a, V_b it is then true that $P_{a,v}$ and $P_{b,v}$ are corresponding points in the birational correspondence π_a^b. Hence $P_v^* = \{P_{a,v}\}$ is a point of M. *Thus every zero-dimensional valuation v determines uniquely a point P_v^* of M.*

If v_1 and v_2 are two distinct zero-dimensional valuations, then there exists at least one projective model V_a such that $P_{a,v_1} \neq P_{a,v_2}$. *Hence if $v_1 \neq v_2$ than $P_{v_1}^* \neq P_{v_2}^*$.*

Now let P^* be an arbitrary point of M, $P^* = \{P_a\}$. We denote by $\mathfrak{B}$ the least ring containing the quotient rings $Q(P_a)$. Let V_b be a fixed projective model of Σ/k and let $P_b = \pi_b^* P^*$. *We assert that if ω is a non-unit in $Q(P_b)$ then* $1/\omega \notin \mathfrak{B}$. For assume that $1/\omega \in \mathfrak{B}$. Then $1/\omega$ will belong to the ring generated by a finite number of quotient rings $Q(P_a)$, say $Q(P_{a_1})$, $Q(P_{a_2})$, $\cdots$, $Q(P_{a_m})$. Let V_c be the join of the varieties V_b, V_{a_1}, V_{a_2}, $\cdots$, V_{a_m} and let $P_c = \pi_c^* P^*$. Since $\pi_{a_i}^* P^* = P_{a_i}$ and $\pi_b^* P^* = P_b$, we have $\pi_{a_i}^c P_c = P_{a_i}$ and $\pi_b^c P_c = P_b$, and hence $Q(P_{a_i}) \subseteq Q(P_c)$, $Q(P_b) \subseteq Q(P_c)$. Therefore $1/\omega \in Q(P_c)$. This is a contradiction since any non-unit of $Q(P_b)$ is obviously also a non-unit in $Q(P_c)$.

We have therefore shown that $\mathfrak{B}$ is a proper ring (not a field). We now show that $\mathfrak{B}$ *is a valuation ring*. For this it is sufficient to show that if ξ is any element of Σ then either $\xi \in \mathfrak{B}$ or $1/\xi \in \mathfrak{B}$. We consider again a fixed projective model V_b of Σ/k. We select a system of nonhomogeneous coördinates $x_1, x_2, \cdots, x_n$ of the general point of V_b in such a fashion that the point P_b $(=\pi_b^* P^*)$ is at finite distance with respect to these coördinates. Let V_d be the projective model whose general point has as nonhomogeneous coördinates the elements $x_1, x_2, \cdots, x_n, \xi$. If the point P_d $(=\pi_d^* P^*)$ is at finite distance with respect to these coördinates, then $\xi \in Q(P_d) \subseteq \mathfrak{B}$. If P_d is a point at infinity, we observe first of all that in the above proof of our assertion $1/\omega \notin \mathfrak{B}$ we have shown incidentally the following: if V_a and V_b are *any* two projective models of Σ/k and if $\pi_a^* P^* = P_a$ and $\pi_b^* P^* = P_b$, then P_a and P_b are *corresponding* points of the birational correspondence between V_a and V_b. For on the join V_c of V_a and V_b we have the point $P_c = \pi_c^* P^*$ and the relations $Q(P_c) \supseteq Q(P_a)$, $Q(P_c) \supseteq Q(P_b)$. These relations show that if v is any zero-dimensional valuation whose center on V_c is the point P_c, then the center of v on V_a is P_a and its center on V_b is P_b. Hence P_a and P_b are indeed corresponding points.[7] With this observation in mind, let v be a zero-dimensional valuation whose center on V_b is the point P_b and whose center on V_d is the point P_d. Since P_b is at finite distance, we have $v(x_i) \geqq 0$, $i=1, 2, \cdots, n$. Since P_d is at infinity, we must have $v(\xi)<0$. Hence $v(1/\xi)>0$, $v(x_i/\xi)>0$, and this shows that if we take $1/\xi, x_1/\xi, \cdots, x_n/\xi$ as nonhomogeneous coördinates of the general point of V_d, then P_d is at finite distance. Hence $1/\xi \in Q(P_d) \subseteq \mathfrak{B}$. This completes the proof of our assertion that $\mathfrak{B}$ is a valuation ring.

Let v be the valuation defined by the valuation ring $\mathfrak{B}$. We assert that *v is zero-dimensional.* For let v be of dimension s. We can find a projective model V_b on which the center of v is an s-dimensional variety W. If $P_b = \pi_b^* P^*$, then $Q(P_b) \subseteq \mathfrak{B}$ and this implies that $P_b \in W$.[8] If $s>0$, then we can find a non-unit ω in $Q(P_b)$ such that $\omega \neq 0$ on W, whence $1/\omega \in Q(W) \subseteq \mathfrak{B}$, a contradiction. Hence $s=0$, as asserted.

The above relation $P_b \in W$ implies now $P_b = W$. This is true for any projective model V_b, that is, the center of v on any projective model V_b is the point $P_b = \pi_b^* P^*$. This completes the proof of the theorem.

3. **A generalization. Infinite direct products of projective lines.** The idea of topologizing an algebraic variety V by choosing as closed sets the algebraic subvarieties of V can be used with good effect in order to topologize the set M^* of all homomorphic mappings of any abstract field A into another abstract field K. In this general case we are dealing essentially with a generalization of the concept of the Riemann manifold of a field of algebraic functions (see the Remark at the end of the paper). We begin with some topological preliminaries.

Let $\{R_a\}$ be a system of compact topological spaces indexed by a set $A = \{a\}$. We assume that each R_a is a T_1-space; that is, that the points of R_a are closed sets. Elements of A shall be denoted by small Latin letters, $a, b, c, \cdots$; subsets of A shall be denoted by small Greek letters, $\alpha, \beta, \gamma, \cdots$. If α is a subset of A we shall denote by R_α the direct product $P_{a \in \alpha} R_a$. If $\alpha \subset \beta$ we denote by π_α^β *the projection* of R_β onto R_α. Finally, elements of R_a and R_α shall be denoted by $x_a, y_a, z_a, \cdots$ and by $x_\alpha, y_\alpha, z_\alpha, \cdots$ respectively. If $a \in \alpha$ and if $\pi_a^\alpha x_\alpha = x_a$, then x_a shall be referred to as the a-component of x_α.

We assume that for each *finite* subset α of A a topology has been assigned to R_α and that the following three conditions are satisfied: (1) *R_α is a compact topological space*; (2) *if $\alpha \subset \beta$ then π_α^β is a closed*

[7] See our definition of corresponding points of a birational transformation, loc. cit. footnote 6, p. 505.

[8] See loc. cit. footnote 6, Theorem 3, p. 497.

mapping (mapping = single-valued continuous transformation); (3) *if* α *is a set with one element a then the topology assigned to* R_α *is exactly the topology of* R_a. It is clear that in virtue of these two conditions R_α is a T_1-space. For if x_a is the a-component of x_α, then $(\pi_a^\alpha)^{-1}x_a$ is closed and x_α is the intersection of the closed sets $(\pi_a^\alpha)^{-1}x_a$, $a \in \alpha$.

If we consider only finite subsets α of A and if we define a partial ordering in the collection $\{R_\alpha\}$ by setting $R_\alpha \prec R_\beta$ if $\alpha \subset \beta$, then we have an inverse system $\{R_\alpha; \pi_\alpha^\beta\}$. It is clear that set-theoretically the limit space R^* of the system coincides with the direct product $R^* = \boldsymbol{P}_{a \in A} R_a$. However, the topology in R^* is not necessarily the usual topology of the product space, for our topology in R^* depends not only on the topology of each factor R_a but also on the topology which has been assigned to each R_α, for α any finite subset of A.

Our space R^* is compact, by Steenrod's theorem. We are dealing here with a special case of Steenrod's theorem, and the proof of the compactness of R^* can be somewhat simplified. For this reason, and also for the convenience of the reader, we shall include here a proof of the compactness of R^*.

We have to show that if a family of closed sets in R^* has the finite intersection property (that is, if every finite subfamily has a non-empty intersection), then the intersection of the entire family is non-empty. It will be sufficient to prove this for families of basic closed sets F_α^*, $F_\alpha^* = \pi_\alpha^{-1} F_\alpha$, where F_α denotes a closed set in R_α and where π_α is the projection of R^* onto R_α. Let then $\{F_\alpha^*\}$ be a family $\mathfrak{F}$ of basic closed sets which has the finite intersection property. By Zorn's lemma the family $\{F_\alpha^*\}$ is contained in a maximal family $\{G_\alpha^*\}$ of basic closed sets which has the finite intersection property. It will be sufficient to show that $\bigcap G_\alpha^*$ is non-empty. We shall therefore assume that our original family $\{F_\alpha^*\}$ is not contained properly in another family of basic closed sets which has the finite intersection property.[9]

We first observe that *the intersection of any finite collection of basic closed sets is itself a basic closed set.* For let $\{\pi_{\alpha_i}^{-1} F_{\alpha_i}\}$ be a finite collection of basic closed sets. We put $\alpha = \bigcup \alpha_i$, $F_\alpha = \bigcap (\pi_{\alpha_i}^\alpha)^{-1} F_{\alpha_i}$. Then it is clear that $\bigcap \pi_{\alpha_i}^{-1} F_{\alpha_i} = \pi_\alpha^{-1} F_\alpha$.

In virtue of this remark and in virtue of the maximality property

[9] The idea of passing to a maximal family is taken from the proof of Tychonoff's theorem as given in Lefschetz, *Algebraic topology*, p. 19. There is only this difference: the maximal family in Lefschetz is not a family of closed sets, while ours is. This modification of the proof succeeds because we restrict ourselves to families of basic closed sets and because in our case the mappings π_α^β are closed.

of the given family $\mathfrak{F}$, it follows that every finite intersection of sets in $\mathfrak{F}$ is again in the family $\mathfrak{F}$.

For any element a in A and for any member F_α^* in $\mathfrak{F}$ let $\pi_a F_\alpha^* = F_{\alpha,a}$. If $a \notin \alpha$ then it is clear that $F_{\alpha,a} = R_a$, for then the a-component of the points of F_α^* is not restricted. If $a \in \alpha$ and if $F_\alpha^* = \pi_\alpha^{-1} F_\alpha$, then $F_{\alpha,a} = \pi_a^\alpha F_\alpha$. In either case $F_{\alpha,a}$ is a closed set in R_a, for we have assumed that π_α^β is closed whenever $\alpha \subset \beta$. For a given a the family $\mathfrak{F}_a$ of closed sets $\{F_{\alpha,a}\}$ has the finite intersection property. Since R_a is compact, the intersection $\bigcap_\alpha F_{\alpha,a}$ is non-empty. Let x_a be a point common to all the sets in $\mathfrak{F}_a$. Then $\pi_a^{-1} x_a$ is a basic closed set (since R_a is a T_1-space) which meets every set F_α^* in $\mathfrak{F}$. Consequently $\pi_a^{-1} x_a \in \mathfrak{F}$, $x_a \in \mathfrak{F}_a$, and the intersection $\bigcap_\alpha F_{\alpha,a}$ consists only of the point x_a.

Let then $x = \{x_a\}$, where $x_a = \bigcap_\alpha F_{\alpha,a}$. We show that x is a common point of the sets F_α^* in $\mathfrak{F}$. Since $\pi_a^{-1} x_a \in \mathfrak{F}$, for any a, it follows that $\bigcap_{a \in \alpha} \pi_a^{-1} x_a \in \mathfrak{F}$, that is, $\pi_\alpha^{-1} x_\alpha \in \mathfrak{F}$, where $x_\alpha = \pi_\alpha x$. Therefore $\pi_\alpha^{-1} x_\alpha$ meets F_α^*, that is, $\pi_\alpha^{-1} F_\alpha$; hence $x_\alpha \in F_\alpha$ and $x \in \pi_\alpha^{-1} x_\alpha \subset \pi_\alpha^{-1} F_\alpha = F_\alpha^*$, q.e.d.

Now let K be a fixed abstract field and let the sets R_a be projective lines over K, so that the points of each set R_a are in (1, 1) correspondence with the elements of K together with the symbol ∞. We topologize R_a by choosing as closed sets the finite subsets of R_a. Then each R_a becomes a compact topological T_1-space.

We still have to topologize each set R_α, for α a finite subset of A. For this purpose we introduce on each line R_a a pair of homogeneous coördinates x_{a1}, x_{a2} and we define an algebraic variety V_α by the following parametric equations (in which the $X_{(\epsilon)}^{(\alpha)}$ denote the homogeneous coördinates of the general point of V_α):

$$(1) \qquad \rho \cdot X_{\epsilon_1 \epsilon_2 \cdots \epsilon_n}^{(\alpha)} = x_{a_1 \epsilon_1} x_{a_2 \epsilon_2} \cdots x_{a_n \epsilon_n},$$

where $\alpha = \{a_1, a_2, \cdots, a_r\}$ and where each ϵ_j can take the values 1 or 2. It is well known that V_α is a Segre variety, of dimension n, immersed in a projective space of dimension $2^n - 1$. The points of V_α are in (1, 1) correspondence with n-tuples of ratios $\{x_{a2}/x_{a1}\}$, $a \in \alpha$, that is, with the points of the direct product $R_\alpha = R_{a_1} \times R_{a_2} \times \cdots \times R_{a_n}$. It should be noted that here we only consider points X^α whose homogeneous coördinates are in K. *We topologize V_α by choosing as closed sets the algebraic subvarieties of V_α.* Then it is clear that each V_α becomes a compact topological T_1-space.

If $\alpha = \{a_1, a_2, \cdots, a_n\}$ and if β is a subset of α, say if $\beta = \{a_1, a_2, \cdots, a_m\}$, $m < n$, then the projection π_β^α of V_α onto V_β is given by the equations:

$$X_{\epsilon_1 \epsilon_2 \cdots \epsilon_m}^{(\beta)} : X_{\delta_1 \delta_2 \cdots \delta_m}^{(\beta)} = X_{\epsilon_1 \epsilon_2 \cdots \epsilon_m \gamma_{m+1} \cdots \gamma_n}^{(\alpha)} : X_{\delta_1 \delta_2 \cdots \delta_m \gamma_{m+1} \cdots \gamma_n}^{(\alpha)},$$

where each ϵ, δ and γ can take *independently* the values 1 or 2. Thus π_β^α is a single-valued *rational* transformation of V_α onto V_β, and therefore π_β^α is closed and $(\pi_\beta^\alpha)^{-1}$ is open. It is clear that the closed sets in the infinite direct product R^*, as defined above, are the sets defined by (finite or infinite) systems of homogeneous equations, each equation involving the variables $X^{(\alpha)}$ relative to some finite subset α of A.

4. The space of homomorphic mappings of one abstract field into another. We now further specialize our application by assuming that the set A is a field. The space R^* is then the space of all single-valued transformations $x^*: a \to x_a = x_{a1}/x_{a2}$, of the field A into the set consisting of the elements of the field K and of the symbol ∞. We shall now express in an appropriate homogeneous form the conditions that a given mapping x^* be a homomorphism. Let α be a subset of A consisting of three elements, $\alpha = \{a_1, a_2, a_3\}$. On the corresponding variety V_α let F_{a_1,a_2,a_3} be the algebraic subvariety obtained by imposing on the 6 parameters x_{i1}, x_{i2}, $i = a_1, a_2, a_3$, the following condition:

$$x_{a_1 1}x_{a_2 2}x_{a_3 2} + x_{a_1 2}x_{a_2 1}x_{a_3 2} = x_{a_1 2}x_{a_2 2}x_{a_3 1}. \tag{2}$$

Similarly we define another algebraic subvariety G_{a_1,a_2,a_3} of V_α by the equation

$$x_{a_1 1}x_{a_2 1}x_{a_3 2} = x_{a_1 2}x_{a_2 2}x_{a_3 1}. \tag{3}$$

Let $x_{a_j 1}/x_{a_j 2} = x_j$, $j = 1, 2, 3$, where x_j may be ∞. Suppose that equation (2) holds true. Then if x_1 and x_2 are both different from ∞ we find $x_3 = x_1 + x_2$. If $x_1 = \infty$ and $x_2 \neq \infty$, then $x_{a_1 2} = 0$, $x_{a_1 1} \cdot x_{a_2 2} \neq 0$, whence (2) yields $x_{a_3 2} = 0$, that is, $x_3 = \infty$. Assume now that equation (3) holds. Again we find that $x_3 = x_1 x_2$, if both x_1 and x_2 are different from ∞. If $x_1 = \infty$ and $x_2 \neq 0$, then $x_{a_1 2} = 0$, $x_{a_1 1} \cdot x_{a_2 1} \neq 0$, and (3) yields $x_{a_3 2} = 0$, that is, $x_3 = \infty$. Thus the equations (2) and (3) are the homogeneous counterparts of the equations $x_3 = x_1 + x_2$ and $x_3 = x_1 x_2$ respectively, and they include the conventions which are usually made for the symbol ∞. We can therefore assert that x^* represents a homomorphic mapping of A into (K, ∞) if and only if the following conditions are satisfied: for any three elements a_1, a_2, a_3 of A such that respectively $a_3 = a_1 + a_2$ or $a_3 = a_1 a_2$, the projection $\pi_\alpha^{-1} x^*$ (where $\alpha = \{a_1, a_2, a_3\}$) must lie respectively on F_{a_1,a_2,a_3} or on G_{a_1,a_2,a_3}. Therefore, if we denote by M the set of all *homomorphic* mappings of A into (K, ∞), we see that

$$M = \bigcap_\alpha \overset{-1}{\pi_\alpha} F_{a_1 a_2 a_3} \bigcap_\beta \overset{-1}{\pi_\beta} G_{b_1 b_2 b_3},$$

where the index α ranges over all sets $\alpha = \{a_1, a_2, a_3\}$ such that $a_3 = a_1 + a_2$, and the index β ranges over the sets $\beta = \{b_1, b_2, b_3\}$ such that $b_3 = b_1 b_2$. We see thus that M is an intersection of basic closed subsets of R^*. Hence M is closed, and since R^* is compact M is also compact.

The case which is of special interest to us is that in which K is a subfield of A. In this case we are interested in the relative homomorphisms of A into (K, ∞), that is, in the homomorphisms x^* which leave each element of K invariant. If M^* is the set of all these relative homomorphisms, then it is clear that M^* is the intersection of M with the closed set $\bigcap_{a \in K} \pi_a^{-1} a$. Here, according to our notations, $\pi_a^{-1} a$ denotes that subset of R^* which consists of the points x^* whose a-component x_a is a itself $(a \in K)$. Hence also M^* is a compact space.

It is convenient to describe in algebro-geometric terms the relative topology induced in M^* by the topology of M. Let $x_1, x_2, \cdots, x_n$ be a finite set of elements of A. For each x_i we introduce a pair of homogeneous parameters x_{i1}, x_{i2} such that $x_{i1}/x_{i2} = x_i$. We consider the algebraic variety Z over K whose general point has as homogeneous coördinates the quantities $X_{(\epsilon)}$ defined by the parametric equations

$$\rho X_{\epsilon_1 \epsilon_2 \cdots \epsilon_n} = x_{1\epsilon_1} x_{2\epsilon_2} \cdots x_{n\epsilon_n}, \tag{4}$$

where each ϵ_j can take the values 1 or 2. If the quantities x_i are algebraically independent, then the variety Z coincides with the variety V_α defined by the equations (1), α being the subset $\{x_1, x_2, \cdots, x_n\}$ of A. But in general Z is a subvariety of V_α. If $x^* \in M^*$, then the mapping x^* of A into (K, ∞) must preserve all the algebraic relations between $x_1, x_2, \cdots, x_n$ over K, since x^* is a homomorphism. It follows that the point $\pi_\alpha x^*$ of V_α must lie on Z. Now we observe that the homomorphism x^* defines a unique valuation of A/K whose residue field is K itself and whose center on Z is the point $\pi_\alpha x^*$. Conversely, every valuation of A/K whose residue field is K defines a relative homomorphic mapping of A/K onto (K, ∞). We conclude that *if W is any algebraic subvariety of Z, then the set of all valuations of A/K having K as residue field and having center on W is a closed subset of M^**. By taking different finite subsets $\{x_1, x_2, \cdots, x_n\}$ of A and different subvarieties W of Z we obtain a family of closed subsets of M^* which form a basis for the closed subsets of M^*.

Remark. Suppose that A is a field of algebraic functions in any number of variables, over a given ground field k. We identify the field K with the algebraically closed field determined by k. The Riemann manifold M of A is the set of all zero-dimensional valuations v

of A. By the ground field extension $k \rightarrow K$ we can embed A in a field $A' = KA$. Every relative homomorphic mapping of A' onto (K, ∞) determines uniquely a zero-dimensional valuation of A'/K, and vice versa. Every zero-dimensional valuation of A'/K induces a unique zero-dimensional valuation of A/K, but a given zero-dimensional valuation of A/K may be extendable in more than one way to a zero-dimensional valuation of A'/K. It follows that the Riemann manifold M' of A'/K coincides with the space M^* of all relative homomorphic mappings of A' onto (K, ∞) and is therefore a compact space. The Riemann manifold M of A/K is obtainable from M' by topological identification and therefore can also be converted into a compact topological space. That is precisely what we have proved in §2.

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