ON LINEAR EQUATIONS IN HILBERT SPACE

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Given an infinite matrix $A = \|a_{ij}\|$ where a_{ij} is complex and

$$\sum_{j=1}^{\infty} | a_{ij} |^2 < + \infty, \qquad i = 1, 2, \cdots, \tag{1}$$

the problem of solving the system of linear equations

$$y_i = \sum_{j=1}^{\infty} a_{ij} x_j, \qquad i = 1, 2, \cdots, \tag{2}$$

has been studied from several points of view. For arbitrary y_i, E. Schmidt[1] has given necessary and sufficient conditions on the a_{ij}, y_i so that the system (2) have a solution $x = (x_j) \in H_2$ (Hilbert space). Schmidt shows that if a solution exists, the solution of minimum norm is unique, and gives explicit formulas for this solution. If A defines a linear transformation T on H_2 to H_2, F. Riesz[2] gives necessary and sufficient conditions that an inverse T^{-1} exist, that is, that the solution $x = T^{-1}(y)$ where T^{-1} is a linear transformation. The following problem stands between these two: Find conditions on the elements of A so that the system (2) have a solution $x \in H_2$ for each $y \in H_2$. Such conditions will permit the use of Schmidt's formulas to express the minimal solution x for each y but this of course does not imply the existence of an inverse of the matrix A. We give a solution of this problem by a method which depends on a property, which seems new, of the m-rowed minors of the matrices $A_{i_1 \cdots i_m} = \|a_{i_s j}\|_{1 \leqq s \leqq m; j \geqq 1}$ and on Cramer's rule.

Let

$$a(i_1, \cdots, i_m; j_1, \cdots, j_m) = \det \|a_{i_s j_t}\|_{1 \leqq s, t \leqq m}$$

be the determinant of the columns $j_1, \cdots, j_m$ of $A_{i_1 \cdots i_m}$. If $B = \|b_{ij}\|$ satisfies (1) and $B'_{i_1 \cdots i_m}$ is the transposed of $B_{i_1 \cdots i_m}$, the determinant $\det A_{i_1 \cdots i_m} B'_{i_1 \cdots i_m} = \det \|\sum_{k=1}^{\infty} a_{i_s k} b_{i_t k}\|_{1 \leqq s, t \leqq m}$ is finite. Because of the continuity of a determinant as a function of its elements

$$\det A_{i_1 \cdots i_m} B'_{i_1 \cdots i_m} = \lim_n \det \left\| \sum_{k=1}^{n} a_{i_s k} b_{i_t k} \right\|_{1 \leqq s, t \leqq m}. \tag{3}$$

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[1] E. Schmidt, *Über die Auflösung linearer Gleichungen mit unendlich vielen Unbekannten*, Rend. Circ. Mat. Palermo vol. 25 (1908) pp. 53–77.

[2] F. Riesz, *Les systèmes d'equations linéaires a une infinité d'inconnues*, Paris, 1913, p. 86.

There is a theorem[3] on the minors of products of square matrices which, with slight modification in its proof, yields the identity

$$\det \left\| \sum_{k=1}^{n} a_{i_s k} b_{i_t k} \right\|_{1 \leqq s, t \leqq m} = \sum_{[j_1 \cdots j_m]}^{n} a(i_1, \cdots, i_m; j_1, \cdots, j_m) b(i_1, \cdots, i_m; j_1, \cdots, j_m), \quad n \geqq m, \tag{4}$$

where the sum is extended over all combinations $j_1, \cdots, j_m$ in $1, \cdots, n$.

THEOREM 1. *If* A, B *satisfy* (1), *then*

$$\det A_{i_1 \cdots i_m} B'_{i_1 \cdots i_m} = \sum_{[j_1 \cdots j_m]}^{\infty} a(i_1, \cdots, i_m; j_1, \cdots, j_m) b(i_1, \cdots, i_m; j_1, \cdots, j_m),$$

where the sum is extended over all combinations of positive integers $j_1, \cdots, j_m$. *The series converges absolutely and*

$$| \det A_{i_1 \cdots i_m} B'_{i_1 \cdots i_m} | \leqq [\det A_{i_1 \cdots i_m} \overline{A}'_{i_1 \cdots i_m}]^{1/2} [\det B_{i_1 \cdots i_m} \overline{B}'_{i_1 \cdots i_m}]^{1/2}.$$

PROOF. By Schwartz' inequality we have, from (4) and (3) with $B = \overline{A}$,

$$\begin{aligned} \sum_{[j_1 \cdots j_m]}^{n} & | a(i_1, \cdots, i_m; j_1, \cdots, j_m) b(i_1, \cdots, i_m; j_1, \cdots, j_m) | \\ & \leqq \left[\sum_{[j_1 \cdots j_m]}^{n} | a(i_1, \cdots, i_m; j_1, \cdots, j_m) |^2 \right]^{1/2} \cdot \left[\sum_{[j_1 \cdots j_m]}^{n} | b(i_1, \cdots, i_m; j_1, \cdots, j_m) |^2 \right]^{1/2} \\ & \leqq [\det A_{i_1 \cdots i_m} \overline{A}'_{i_1 \cdots i_m}]^{1/2} [\det B_{i_1 \cdots i_m} \overline{B}'_{i_1 \cdots i_m}]^{1/2}, \qquad n \geqq m. \end{aligned}$$

The conclusion is now evident as a consequence of (3).

If we define

$$| AB' | = \lim_m \sup \sup_{i_1 \cdots i_m} \left| \sum_{[j_1 \cdots j_m]}^{\infty} a(i_1, \cdots, i_m; j_1, \cdots, j_m) x \times b(i_1, \cdots, i_m; j_1, \cdots, j_m) \right|$$

[3] C. C. MacDuffee, *An introduction to abstract algebra*, New York, 1940, Theorem 99.1, p. 216.

we have a Schwarz inequality for matrices:

$$|AB'| \leqq |A\bar{A}'|^{1/2} |B\bar{B}'|^{1/2}.$$

The following lemma contains the Gram condition for linear dependence.

LEMMA 1. *The following statements are equivalent:*
(a) *The rows of $A_{1\cdots m}$ are linearly dependent.*
(b) Det $A_{1\cdots m}\bar{A}'_{1\cdots m} = 0$.
(c) *All m-rowed minors of $A_{1\cdots m}$ equal zero.*

PROOF. That (a) implies (c) is immediate. The equivalence of (b) and (c) follows from Theorem 1 with $A = B$. It remains to show that (c) implies (a). This is evident if $m = 1$. Assuming this statement for $m-1$, it is true for m if all the $(m-1)$-rowed minors of A_{m-1} vanish. If one such minor does not vanish, say

$$\det \| a_{ij} \|_{1 \leqq i, j \leqq m-1} \neq 0,$$

we denote by c_k the cofactor of a_{km} in the determinant of the first m columns of $A_{1\cdots m}$. Then $c_m \neq 0$ and

$$\sum_{k=1}^{m} c_k a_{kj} = 0, \qquad j = 1, \cdots, m-1.$$

But this sum vanishes for all other values of j because of (c). Hence (c) implies (a).

THEOREM 2. *If A satisfies* (1), *the finite system*

$$y_i = \sum_{j=1}^{\infty} a_{ij} x_j, \qquad i = 1, \cdots, m, \tag{5}$$

has a solution $x \in H_2$ for each $y_1, \cdots, y_m$ if and only if

$$\det A_{1\cdots m}\bar{A}'_{1\cdots m} \neq 0.$$

PROOF. The necessity is a consequence of Lemma 1. If the condition is satisfied, then there is a nonvanishing $a(1, \cdots, m; j_1, \cdots, j_m)$ by (c) of Lemma 1 and a solution $x = (x_j)$ where $x_j = 0$ for $j \neq j_1, \cdots . j_m$ and $x_{j_1}, \cdots, x_{j_m}$ are determined by Cramer's rule.

COROLLARY. *If A satisfies* (1) *and the system* (2) *has a solution $x \in H_2$ for each $y \in H_2$, then* det $A_{1\cdots m}\bar{A}'_{1\cdots m} \neq 0$ *for all m.*

An estimate of the minimum norm of the solution of the finite system (5) may be given in terms of a series of finite minors in $A_{i_1\cdots i_m}$.

Let $J=[j_1, \cdots, j_m]$ be a combination of m distinct positive integers and let S_m be a set of J such that no two J's have a common integer while every positive integer is in some $J \in S_m$. Let

$$\alpha_{i_1 \cdots i_m} = \left[\sup_{S_m} \sum_{J \in S_m} | a(i_1, \cdots, i_m; j_1, \cdots, j_m |^2 \right]^{1/2},$$

$$a_{i_1 \cdots i_m} = \left[\sum_{[j_1 \cdots j_m]} | a(i_1, \cdots, i_m; j_1, \cdots, j_m |^2 \right]^{1/2}.$$

Since S_m is a subset of the sum of all $j_1, \cdots, j_m$ we have the following lemma.

LEMMA 2. $\alpha_{i_1 \cdots i_m} \leqq a_{i_1 \cdots i_m}$,

$$\sup_{S_m} \sum_{J \in S_m} \sum_{s=1}^{m} |a(1, \cdots, k-1, k+1, \cdots, m; j_1, \cdots, j_{s-1} j_{s+1}, \cdots, j_m)|^2$$
$$\leqq \sum_{[j_1 \cdots j_{m-1}]}^{\infty} | a(1, \cdots, k-1, k+1, \cdots, m; j_1, \cdots, j_{m-1})|^2.$$

THEOREM 3. *If A satisfies (1) and the finite system (5) has a solution $x^m \in H_2$ for each $y^m = (y_1, \cdots, y_m, 0, 0, \cdots)$, then*

$$\inf \|x^m\| \leqq \|y^m\| \left[\sum_{k=1}^{m} \left| \frac{a_{1, \cdots, k-1, k+1, \cdots, m}}{\alpha_{1 \cdots m}} \right|^2 \right]^{1/2}.$$

PROOF. By Theorem 2, det $A_{1 \cdots m} \bar{A}'_{1 \cdots m} \neq 0$ and so $\alpha_{1 \cdots m} \neq 0$. Let $M^m_{kj_s}$ be the cofactor of a_{kj_s} in $a(1, \cdots, m; j_1, \cdots, j_m)$. The system (5) has a solution $x^m_{j_1 \cdots j_m}$ defined by

$$x^m_{j; j_1 \cdots j_m} = \begin{cases} \sum_{k=1}^{m} y_k M^m_{kj} / a(1, \cdots, m; j_1, \cdots, j_m), & j = j_1, \cdots, j_m, \\ 0, & j \neq j_1, \cdots, j_m. \end{cases}$$

We have

$$\|x^m_{j_1 \cdots j_m}\|^2 | a(i_1, \cdots, i_m; j_1 \cdots j_m)|^2$$
$$\leqq \sum_{s=1}^{m} \left| \sum_{k=1}^{m} y_k M^m_{kj_s} \right|^2 \leqq \|y^m\|^2 \sum_{s=1}^{m} \sum_{k=1}^{m} | M^m_{kj_s}|^2$$
$$= \|y^m\|^2 \sum_{s=1}^{m} \sum_{k=1}^{m} | a(1, \cdots, k-1, k+1, \cdots, m; j_1, \cdots, j_{s-1}, j_{s+1}, \cdots, j_m)|^2.$$

Hence

$$\inf \|x^m\|^2 \alpha_{1\cdots m}^2 \leqq \|y^m\|^2 \sum_{k=1}^{m} \sup_{S_m} \sum_{J\in S_m} \sum_{s=1}^{m} | a(1, \cdots, k-1, k+1, \cdots, m; j_1 \cdots j_{s-1} j_{s+1} \cdots j_m) |^2$$
$$\leqq \|y^m\|^2 \sum_{k=1}^{m} a_{1,\cdots,k-1,k+1,\cdots,m}^2$$

by Lemma 2. The conclusion follows at once.

A sufficient condition for the solution of the system (2) for each $y\in H$ may be obtained by restricting the constants

$$\alpha_m = \left[\sum_{k=1}^{m} \left| \frac{a_{1\cdots k-1,k+1,\cdots,m}}{\alpha_{1\cdots m}} \right|^2 \right]^{1/2}, \qquad m = 1, 2, \cdots .$$

THEOREM 4. *If A satisfies* (1), *its rows are linearly independent, and* $\alpha = \lim \inf_m \alpha_m < +\infty$, *then for each $y\in H_2$ the system* (2) *has a solution $x\in H_2$ such that*

$$\|x\| \leqq \alpha\|y\|.$$

PROOF. Consider any $y\in H_2$ and any $\epsilon>0$. The sequence contains a subsequence $\alpha_{m_\mu} < \alpha+\epsilon$. From Theorem 3 it follows that for each μ there is an $x^\mu = (x_j^\mu) \in H_2$ such that

$$y_i = \sum_{j=1}^{\infty} a_{ij} x_j^\mu, \qquad i = 1, \cdots, m_\mu,$$
$$\|x^\mu\| \leqq (\alpha + \epsilon)\|y\|.$$

Applying a diagonal process to (x_j^μ) one finds a subsequence $x^{\mu_\nu} = (x_j^{\mu_\nu})$ and an $x = (x_j) \in H_2$ such that

$$\lim_\nu x_j^{\mu_\nu} = x_j, \qquad j = 1, 2, \cdots ,$$
$$\|x\| \leqq (\alpha + \epsilon)\|y\|.$$

Since for all ν, $N>0$ and $1 \leqq i \leqq m_{\mu_\nu}$

$$y_i - \sum_{j=1}^{\infty} a_{ij}x_j = \sum_{j=1}^{N} a_{ij}(x_j^{\mu_\nu} - x_j) + \sum_{j=N+1}^{\infty} a_{ij}x_j^{\mu_\nu} - \sum_{j=N+1}^{\infty} a_{ij}x_j,$$

x solves the system (1).

Now consider $\epsilon_n \downarrow 0$. For each n there is an $x^n \in H_2$ which solves the system (2) and such that $\|x^n\| \leqq (\alpha+\epsilon_n)\|y\|$. Repeating the diagonal process and the above argument, one finds an $x\in H_2$ such that $\|x\| \leqq \alpha\|y\|$ and which solves the system (2).

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