THE CONVERGENCE AND VALUES OF PERIODIC CONTINUED FRACTIONS

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1. Introduction. Much of the recent progress in the study of continued fractions is due to the consideration of the continued fraction as a product of linear fractional transformations, as opposed to the older approach in which attention is focused on the numerators and denominators, A_n and B_n, of the approximants.

The use of the transformation point of view, besides simplifying the proofs of theorems, often sheds additional light on the significance of the results. This advantage is particularly noticeable in the case of the periodic continued fraction, as the following theorem and its proof will show. The essential portions of the theorem were given by Stolz [1].[1] An improved but lengthy proof was later given by Pringsheim [2]. This was followed by Perron's shorter proof [3], in which ideas related to the transformation point of view were virtually suppressed. The proof given here makes full use of linear fractional transformations and is even shorter than Perron's proof.

THEOREM. *Consider the k-term periodic continued fraction*

$$\frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots + \frac{a_k}{b_k} + \frac{a_1}{b_1} + \cdots, \tag{1.1}$$

for which $a_1 a_2 \cdots a_k \neq 0$. *Let* F_ν *denote the* ν*th approximant, so that* $F_\nu = A_\nu / B_\nu$. *Let* S *denote the linear fractional transformation*

$$z' = \frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots + \frac{a_k}{b_k} + \frac{z}{1}, \tag{1.2}$$

and let x_1 *and* x_2 *be the fixed points of* S. *The continued fraction converges if and only if* x_1 *and* x_2 *are finite numbers satisfying one of the following two conditions:*

$$x_1 = x_2 \tag{1.3}$$

or

$$F_r \neq x_2,\ r = 0, 1, 2, \cdots, k-1, \text{ and } |F_{k-1} - x_1| < |F_{k-1} - x_2|. \tag{1.4}$$

If the continued fraction converges, its value is x_1. *Furthermore,* $F_{nk+r} = x_1$ *for some* r, $0 \leqq r \leqq k-1$, *and for* $n = 1, 2, 3, \cdots$, *if and only if* $F_r = x_1$.

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[1] Numbers in brackets refer to the Bibliography at the end of the paper.

2. Two lemmas. The theorem will be proved by means of two lemmas on linear fractional transformations.

Let T denote the nonsingular linear fractional transformation

$$z' = \frac{az - b}{cz - d}, \qquad bc - ad \neq 0. \tag{2.1}$$

Let x_1 and x_2 be the fixed points of T. A necessary and sufficient condition that both x_1 and x_2 be finite is that $c \neq 0$.

Suppose that $c \neq 0$, and let $f = T(\infty)$. Obviously f is not a fixed point of T; hence f is finite and distinct from x_1 and x_2. Without loss in generality we can require that $|f - x_1| \leqq |f - x_2|$. Furthermore, if x is a fixed point of T, then

$$z' - x = \frac{az - b}{cz - d} - \frac{ax - b}{cx - d} = \frac{bc - ad}{cx - d} \cdot \frac{z - x}{cz - d}.$$

But if x is a fixed point of T, it is also a fixed point of T^{-1}, so that

$$cx - d = c\left(\frac{dx - b}{cx - a}\right) - d = -\frac{bc - ad}{cx - a},$$

or

$$a - cx = \frac{bc - ad}{cx - d}.$$

Hence

$$z' - x = (a - cx)\frac{z - x}{cz - d}. \tag{2.2}$$

This equation will be used in the proof of each of the two lemmas.

LEMMA 2.1. *If $x_1 = x_2$, where x_1 is finite, then T^n can be written*

$$\frac{1}{z' - x_1} = \frac{1}{z - x_1} + \frac{n}{f - x_1}.$$

PROOF. From (2.2) we have

$$\frac{1}{z' - x_1} = \frac{1}{a - cx_1} \cdot \frac{cz - d}{z - x_1} = \frac{1}{a - cx_1}\left(\frac{cx_1 - d}{z - x_1} + c\right).$$

But if T has only one fixed point, x_1, then $x_1 = (a+d)/2c$, so that $2cx_1 = a + d$, or $cx_1 - d = a - cx_1$. Hence

$$\frac{1}{z' - x_1} = \frac{1}{z - x_1} + \frac{c}{a - cx_1}.$$

But $f = T(\infty) = a/c$. Hence T can be written

$$\frac{1}{z' - x_1} = \frac{1}{z - x_1} + \frac{1}{f - x_1}.$$

Upon repeated application of this formula, Lemma 2.1 follows at once.

LEMMA 2.2. *If x_1 and x_2 are finite and distinct, then T^n can be written*

$$\frac{z' - x_1}{z' - x_2} = \left(\frac{f - x_1}{f - x_2}\right)^n \frac{z - x_1}{z - x_2}.$$

PROOF. From (2.2) we obtain by division the equation

$$\frac{z' - x_1}{z' - x_2} = \frac{a - cx_1}{a - cx_2} \cdot \frac{z - x_1}{z - x_2}.$$

But $f = T(\infty) = a/c$. Hence T can be written

$$\frac{z' - x_1}{z' - x_2} = \frac{f - x_1}{f - x_2} \cdot \frac{z - x_1}{z - x_2}.$$

Upon repeated application of this formula, Lemma 2.2 follows at once.

3. **Proof of the theorem.** We observe that the right-hand member of (1.2) can be regarded as a terminating continued fraction for which $a_{k+1} = z$ and $b_{k+1} = 1$, so that S can be written

$$z' = \frac{A_k + zA_{k-1}}{B_k + zB_{k-1}}. \tag{3.1}$$

This observation immediately enables us to obtain the following conclusions:

(i) S is nonsingular; for the value of its determinant is $A_kB_{k-1} - A_{k-1}B_k = (-1)^{k-1}a_1a_2 \cdots a_k$, which is nonvanishing by hypothesis.

(ii) $F_{k-1} = S(\infty)$; consequently if x_1 and x_2 are finite, it follows that F_{k-1}, the transform of the point at infinity, is finite and not a fixed point of S.

(iii) $F_{nk+r} = S^n(F_r)$, where (as in the remainder of this proof) $n = 1, 2, \cdots$, and r is limited to the k values $0, 1, 2, \cdots, k-1$.

(iv) From (iii) it follows that $F_{nk+r} = x_1$ if and only if $F_r = x_1$; consequently no approximant of (1.1) actually takes on the value x_1 unless one of the first k approximants does so.

The rest of the theorem will be proved by considering separately four cases which are classified by the nature of the fixed points of S.

Case I. *The point at infinity a fixed point of* S. From (3.1) we see that this case occurs only if $B_{k-1}=0$. But if $B_{k-1}=0$, then $A_{k-1}\neq 0$, since S is nonsingular. Hence $F_{k-1}=\infty$. Hence $F_{nk+k-1}=\infty$ for all n, so that (1.1) diverges. We remark in passing, however, that if $B_{k-1}=0$, $A_k\neq 0$, and $|A_{k-1}|>|B_k|$, then the reciprocal of (1.1) converges to the value zero.

Case II. $x_1=x_2$, *where* x_1 *is finite.* In this case, Lemma 2.1 applies, so that observation (iii) above enables us to write

$$\frac{1}{F_{nk+r}-x_1}=\frac{1}{F_r-x_1}+\frac{n}{F_{k-1}-x_1}\cdot$$

Since F_{k-1} and x_1 are finite, the right-hand member of this equation can be made arbitrarily large in absolute value by taking n sufficiently large. But if the reciprocal of $|F_{nk+r}-x_1|$ is arbitrarily large, then $|F_{nk+r}-x_1|$ itself is arbitrarily small. Hence $\lim_{n\to\infty} F_{nk+r}=x_1$. It has already been remarked that $F_{nk+r}=x_1$ if and only if $F_r=x_1$.

Case III. x_1 *and* x_2 *distinct finite numbers,* $|F_{k-1}-x_1|<|F_{k-1}-x_2|$. In this case, Lemma 2.2 applies, so that observation (iii) enables us to write

$$\frac{F_{nk+r}-x_1}{F_{nk+r}-x_2}=\left(\frac{F_{k-1}-x_1}{F_{k-1}-x_2}\right)^n\frac{F_r-x_1}{F_r-x_2}=K^n\frac{F_r-x_1}{F_r-x_2},\tag{3.2}$$

where $|K|<1$ by hypothesis, so that $|K^n|$ can be made arbitrarily small by taking n sufficiently large. Hence if $F_r\neq x_2$, the entire right-hand member of (3.2) can be made arbitrarily small in absolute value. If it is made less than unity, then $|F_{nk+r}-x_1|=\epsilon_n|F_{nk+r}-x_2|$, or $|F_{nk+r}-(x_1-\epsilon_n^2x_2)/(1-\epsilon_n^2)|=\epsilon_n|x_1-x_2|/(1-\epsilon_n^2)$, where $0<\epsilon_n<1$, and $\lim_{n\to\infty}\epsilon_n=0$. It follows that $\lim_{n\to\infty}F_{nk+r}=x_1$. Hence if $F_r\neq x_2$ for each of the k values of r, then $\lim_{n\to\infty}F_{nk+r}=x_1$ for all r, so that (1.1) converges to the value x_1.

Suppose, however, that $F_r=x_2$ for one or more of the k values of r; then $F_{nk+r}=x_2$ for those values of r, since x_2 is a fixed point of S. Consequently, x_2 is a limit point of the sequence of approximants of (1.1). But F_{k-1} is not a fixed point of S, as has been pointed out in observation (ii). Consequently $F_{k-1}\neq x_2$, whence $\lim_{n\to\infty}F_{nk+k-1}=x_1$, so that x_1 is a limit point of the sequence of approximants of (1.1). Since $x_1\neq x_2$ by hypothesis, the sequence of approximants has two distinct limit points. Hence (1.1) diverges by oscillation.

Case IV. x_1 *and* x_2 *distinct and finite,* $|F_{k-1}-x_1|=|F_{k-1}-x_2|$. In this case (3.2) holds, where $K=e^{i\theta}$, $K\neq 1$. Since θ is not an integral multiple of 2π, it follows that the quantities K^n have at least two distinct values and do not approach a limit, finite or infinite.

But if the approximants F_{nk+k-1} approach a limit as n increases without limit, then the left-hand member of (3.2) must approach a limit, finite or infinite, as n increases, when $r=k-1$; and since the K^n do not approach a limit, this is possible only if $F_{k-1}=x_1$ or $F_{k-1}=x_2$, which is impossible since F_{k-1} is not a fixed point of S. Hence the sequence of approximants of (1.1) has at least two distinct limit points. The fraction (1.1) therefore diverges by oscillation.

This completes the proof of the theorem.

Bibliography

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3. O. Perron, *Über the Konvergenz periodischer Kettenbrüche*, Sitzungsberichte der mathematisch-physikalischen Klasse der K. Bayerische Akademie der Wissenschaften zu München vol. 35 (1905) pp. 495–503. See also *Die Lehre von den Kettenbrüchen*, Berlin, 1913, p. 272.

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