NOTE ON THE PRECEDING PAPER

IRVING KAPLANSKY AND HARRY POLLARD

The sufficiency portion of the theorem on the harmonic series proved by Erdös and Niven in the preceding paper hinges on the fact that (in their notation) $k_2=k$ implies $k_j=k$ for $j>2$. We shall show that this is true more generally for any series $\sum u_n$ such that $\{u_n\}$ is completely monotonic. The result follows at once from the theorem below.

In the case $k_2>k$, the method has thus far not yielded any result of the kind obtained by Erdös and Niven.

THEOREM. *Let* $u_n \neq 0$ $(n=1, 2, \cdots)$ *be a sequence such that*

$$(1) \qquad (-1)^k \Delta^k u_n \geqq 0 \qquad (k = 0, 1, \cdots ; n = 1, 2, \cdots),$$

that is, $\{u_n\}$ *is completely monotonic, and*

$$(2) \qquad \lim_{n\to\infty} u_{n+1}/u_n = 1.$$

Define

$$S(n, k) = u_n + u_{n+1} + \cdots + u_{n+k-1},$$
$$f(n, k) = S(n + k, k + 1) - S(n, k).$$

Then $f(n, k)>0$ *implies* $f(n+1, k)>0$.

We require the following lemma, which is a consequence of a theorem of D. V. Widder.[1]

LEMMA. *Let* $\phi(t)$ *be a function continuous in* (0, 1) *and having at most one change of sign in this interval. If* $\alpha(t)$ *is non-decreasing in* (0, 1), *then the sequence* v_n *defined by*

$$v_n = \int_0^1 t^n \phi(t) d\alpha(t), \qquad n = 1, 2, \cdots,$$

has at most one change of sign.

PROOF. If $\phi(t)$ is of constant sign in (0, 1) there is nothing to prove. Suppose then that it changes sign at $t=t_0$. Define $\psi(t)=\int_{t_0}^t \phi(t)d\alpha(t)$. Then $\psi(t)$ has at most one change of trend[2] in (0, 1). Since

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[1] D. V. Widder, *The inversion of the Laplace integral and the related moment problem*, Trans. Amer. Math. Soc. vol. 36 (1934) p. 195.

[2] Loc. cit. p. 155.

$v_n = \int_0^1 t^n d\psi(t)$, the result follows from the theorem of Widder.[1]

PROOF OF THE THEOREM. A well known result of Hausdorff states that a sequence satisfying (1) admits the representation

$$u_n = \int_0^1 t^n d\alpha(t),$$

where $\alpha(t)$ is non-decreasing.[3] It follows that

$$f(n, k) = \int_0^1 t^n \phi(t) d\alpha(t),$$

where $\phi(t) = (t^{2k+1} - 2t^k + 1)/(t-1)$. It is readily verified that $\phi(t)$ has exactly one change of sign in (0, 1); hence the same is true for the function $\phi(t) - \epsilon$, for sufficiently small $\epsilon > 0$, say $\epsilon < \epsilon_0$. Thus by the lemma the sequence

$$f(n, k) - \epsilon u_n = \int_0^1 t^n [\phi(t) - \epsilon] d\alpha(t)$$

has at most one change of sign for $\epsilon < \epsilon_0$.

Suppose now that $f(n, k) > 0$, while $f(n+1, k) \leqq 0$. From (2) it follows that $f(N, k) > 0$ for a large enough $N > n+1$. Choose ϵ_1, $0 < \epsilon_1 < \epsilon_0$, so small that

$$f(n, k) - \epsilon_1 u_n > 0,$$
$$f(n + 1, k) - \epsilon_1 u_{n+1} < 0,$$
$$f(N, k) - \epsilon_1 u_N > 0.$$

Then the sequence $f(n, k) - \epsilon_1 u_n$ has at least two changes of sign. But this contradicts the remark above that it can have at most one change of sign. This completes the proof.

NEW YORK CITY

[3] Loc. cit. p. 191.