# CONTINUED FRACTION EXPANSIONS FOR FUNCTIONS WITH POSITIVE REAL PARTS

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1. **Introduction.** Let $K$ denote the region of the complex $z$-plane exterior to the cut along the real axis from $-1$ to $-\infty$. Let $E$ denote the class of functions $F(z)$ with the following three properties:

$$\begin{aligned} &\text{(a)} \quad F(z) \text{ is analytic over } K; \\ &\text{(b)} \quad F(0) = 1; \\ &\text{(c)} \quad R(F(z)) > 0 \text{ over } K. \end{aligned} \tag{1.1}$$

The object of this paper is to prove that the class $E$ is coextensive with the class of functions representable in the form

$$\cfrac{(1+z)^{1/2}}{1+\cfrac{g_1 z}{1+ir_1(1+z)^{1/2}+\cfrac{(1-g_1)g_2 z}{1+ir_2(1+z)^{1/2}+\cfrac{(1-g_2)g_3 z}{1+ir_3(1+z)^{1/2}+\cdots}}}}, \tag{1.2}$$

where $0<g_p<1$, $-\infty<r_p<+\infty$, $p=1, 2, 3, \cdots$, or as a terminating continued fraction of this form, in which the last $g_p$ which appears may be equal to unity. The continued fractions converge uniformly over every bounded closed region within $K$. That branch of $(1+z)^{1/2}$ is to be taken in $K$ which equals 1 for $z=0$.

This result supplements [3].[1] In fact, the continued fraction (1.2) is actually the continued fraction (3.6) of [3]. At that time we did not recognize that the latter can be put in the form (1.2), and we proved convergence only in the neighborhood of the origin. If $r_p=0$, $p=1, 2, 3, \cdots$, the continued fraction (1.2) reduces to a familiar form first considered by E. B. Van Vleck [2], and recently by the present writer [4] in connection with totally monotone sequences. From one point of view, the result is a reformulation of a theorem of Schur [1] on bounded analytic functions.

Presented to the Society, September 17, 1945, under the title *Analytic functions with positive real parts*; received by the editors July 9, 1945.

[1] Numbers in brackets refer to the Bibliography at the end of the paper.

2. **Preliminaries.** It will be convenient to introduce, along with the class $E$, two other classes of analytic functions, $U$ and $V$. The class $U$ consists of all functions $f(w)$ which are analytic and have moduli not greater than unity for $|w|<1$. The class $V$ consists of all functions $k(w)$ which are analytic and have positive real parts for $|w|<1$, and which have the value 1 for $w=0$. The transformation

$$k(w) = \frac{1 + wf(w)}{1 - wf(w)} \tag{2.1}$$

maps $U$ one-to-one upon $V$. We now map the domain $|w|<1$ upon $K$ by means of the transformation

$$z = \frac{4w}{(1-w)^2}, \qquad w = \frac{(1+z)^{1/2} - 1}{(1+z)^{1/2} + 1}. \tag{2.2}$$

The class $E$ then consists of the functions

$$F(z) = k\left(\frac{(1+z)^{1/2} - 1}{(1+z)^{1/2} + 1}\right),$$

where $k(w)$ is in $V$. The correspondence set up in this way between $E$ and $V$ is one-to-one.

To each function $f(w)$ of the class $U$, Schur [1] makes correspond uniquely a finite or infinite sequence of constants $\alpha_p$ such that, in case the sequence is finite and has $n+2$ terms,

$$|\alpha_p| < 1, \quad p = 0, 1, 2, \cdots, n, \quad |\alpha_{n+1}| = 1, \tag{2.3}$$

and, in case the sequence is infinite,

$$|\alpha_p| < 1, \qquad p = 0, 1, 2, \cdots. \tag{2.4}$$

Conversely, each such sequence determines uniquely a function in the class $U$. The correspondence is set up in the following way. If $f(w)$ is any function in $U$, the sequence $\{\alpha_p\}$ is determined recurrently by means of the formulas

$$f_0(w) = f(w), \quad f_{k+1}(w) = \frac{1}{w}\,\frac{\alpha_k - f_k(w)}{1 - \bar{\alpha}_k f_k(w)}, \qquad \alpha_k = f_k(0), \quad k = 0, 1, 2, \cdots. \tag{2.5}$$

Conversely, if a sequence $\{\alpha_p\}$ satisfying (2.3) or (2.4) is given, we construct the linear transformations

$$s = s_p(w; t) = \frac{\alpha_p - wt}{1 - \bar{\alpha}_p wt} \tag{2.6}$$

of the $t$-plane into the $s$-plane, the transformations depending upon the parameter $w$. These have the property that if $|w|<1$ then $|t|\leqq 1$ implies $|s|<1$. The product of two or more of the transformations must have the same property. Thus, if

$$S_p(w;\,t) = s_0s_1\cdots s_p(w;\,t),$$

then

$$|S_p(w;\,t)| < 1 \quad \text{for} \quad |w| < 1 \quad \text{and} \quad |t| \leqq 1. \tag{2.7}$$

To any finite sequence $\{\alpha_p\}$ satisfying (2.3), there corresponds the function $S_n(w;\,\alpha_{n+1})$ in the class $U$. To any infinite sequence $\{\alpha_p\}$ satisfying (2.4) there corresponds the function $f(w)$ in $U$, defined for $|w|<1$ by

$$f(w) = \lim_{p=\infty} S_p(w;\,t_{p+1}). \tag{2.8}$$

Here $t_1, t_2, t_3, \cdots$ is any sequence of numbers with moduli not greater than unity; the limit exists *uniformly* for $|w|\leqq r$ for every positive constant $r$ less than 1; the value of the limit does not depend upon the particular sequence $\{t_p\}$ which is chosen. This formulation differs from that of Schur only in the introduction of the arbitrary sequence $\{t_p\}$, $|t_p|\leqq 1$, which, moreover, entails no essential modification in the proof.

3. **The main theorem.** Let $\alpha_0, \alpha_1, \alpha_2, \cdots, \alpha_p$ be numbers with moduli less than unity, and let

$$f_{k+1}(w;\,t) = \frac{1}{w}\,\frac{\alpha_k - f_k(w;\,t)}{1-\bar{\alpha}_k f_k(w;\,t)}, \quad f_k(w;\,t) = \frac{\alpha_k - wf_{k+1}(w;\,t)}{1-\bar{\alpha}_k w f_{k+1}(w;\,t)}, \tag{3.1}$$
$$k = 0, 1, 2, \cdots, p,$$

where, in the notation introduced before, $f_0(w;\,t) = S_p(w;\,t)$, so that $f_{p+1}(w;\,t) = t$. We introduce functions $h_k(w;\,t)$ by means of the equations

$$h_k(w;\,t) = \frac{1-\beta_k f_k(w;\,t)}{1+w\beta_k f_k(w;\,t)}, \qquad k = 0, 1, 2, \cdots, p+1, \tag{3.2}$$

where the $\beta_k$ are numbers to be determined. If we substitute the values of $f_k(w;\,t)$ and $f_{k+1}(w;\,t)$ obtained from (3.2) into (3.1) we obtain the following formula expressing $h_k = h_k(w;\,t)$ in terms of $h_{k+1} = h_{k+1}(w;\,t)$:

$$h_k = \frac{(\beta_k - \bar{\alpha}_k + \beta_{k+1}\beta_k\alpha_k - \beta_{k+1})wh_{k+1} + (\beta_{k+1}\beta_k\alpha_k - \beta_{k+1} - w\beta_k + w\bar{\alpha}_k)}{(w\bar{\alpha}_k + w^2\beta_k - \beta_{k+1} - w\beta_{k+1}\beta_k\alpha_k) - (\bar{\alpha}_k + w\beta_k + \beta_{k+1} + w\beta_{k+1}\beta_k\alpha_k)wh_{k+1}}.$$

We now determine the $\beta_m$ so that the factor multiplying $h_{k+1}$ in the numerator is zero. This will be true if

$$(3.3)\qquad \beta_k = \frac{\beta_{k-1} - \bar{\alpha}_{k-1}}{1 - \beta_{k-1}\alpha_{k-1}}, \qquad k = 1, 2, 3, \cdots, p+1, \beta_0 = 1.$$

We note that $|\beta_k| = 1$, $k = 0, 1, 2, \cdots, p+1$. With these values of the $\beta_m$, the above formula may be written:

$$(3.4)\ h_k(w; t) = \frac{|1 - \alpha_k\beta_k|^2}{(1 - \bar{\alpha}_k\bar{\beta}_k) - (1 - \alpha_k\beta_k)w + (1 - |\alpha_k\beta_k|^2)wh_{k+1}(w; t)}.$$

In particular, since $\beta_0 = 1$, $f_0(w; t) = S_p(w; t)$, we have

$$(3.5)\quad \frac{1 - S_p(w; t)}{1 + wS_p(w; t)} = \frac{|1 - \alpha_0\beta_0|^2}{(1 - \bar{\alpha}_k\bar{\beta}_k) - (1 - \alpha_0\beta_0)w + (1 - |\alpha_0\beta_0|^2)wh_1(w; t)}.$$

On multiplying both members of (3.5) by $2w/(1-w)$, adding 1 to both sides, and then taking reciprocals, we obtain

$$(3.6)\quad \frac{1 + wS_p(w; t)}{1 - wS_p(w; t)} = \cfrac{1 + w}{1 - w + \cfrac{2w|1 - \alpha_0\beta_0|^2}{(1 - \bar{\alpha}_k\bar{\beta}_k) - (1 - \alpha_0\beta_0)w + (1 - |\alpha_0\beta_0|^2)wh_1(w; t)}}.$$

Let

$$(3.7)\qquad \alpha_k\beta_k = 1 - 2u_{k+1}, \qquad k = 0, 1, 2, \cdots, p.$$

Since $|\beta_k| = 1$, $|\alpha_k| < 1$, then

$$(3.8)\qquad |u_k - 1/2| < 1/2, \qquad k = 1, 2, 3, \cdots, p+1.$$

Thus, $R(u_k) > 0$. We now put

$$(3.9)\qquad g_k = |u_k|^2/R(u_k), \qquad r_k = -I(u_k)/R(u_k).$$

Making the substitutions (3.7) and (2.2) in (3.6) and (3.4), we obtain:

$$(3.10)\quad \frac{1 + wS_p(w; t)}{1 - wS_p(w; t)} = \cfrac{(1 + z)^{1/2}}{1 + \cfrac{g_1 z}{1 + ir_1(1 + z)^{1/2} + (1 - g_1)((1 + z)^{1/2} - 1)h_1(w; t)}},$$

and

$$(3.11)\ h_k(w;t)=\frac{g_{k+1}((1+z)^{1/2}+1)}{1+ir_{k+1}(1+z)^{1/2}+(1-g_{k+1})((1+z)^{1/2}-1)h_{k+1}(w;t)},\quad k=1,2,3,\cdots,p.$$

By (3.8) and (3.9), the numbers $g_k$, $r_k$ satisfy the inequalities

$$(3.12)\qquad 0<g_k<1,\quad -\infty<r_k<+\infty,\qquad k=1,2,3,\cdots,p+1.$$

Conversely, if $g_k$, $r_k$ are any numbers satisfying (3.12), then numbers $u_k$ satisfying (3.8) are uniquely determined by (3.9), inasmuch as the latter may be written $u_k=g_k\cos\phi_k$, $\arg u_k=-\text{arc}\tan r_k=-\phi_k$, $k=1,2,3,\cdots,p+1$. Then, numbers $\alpha_k$ with moduli less than unity are uniquely determined such that (3.3) and (3.7) hold. In fact,

$$\alpha_0=1-2u_1,\qquad \alpha_k=\frac{u_1u_2\cdots u_k}{\bar{u}_1\bar{u}_2\cdots\bar{u}_k}(1-2u_{k+1}),\quad k=1,2,3,\cdots,p.$$

We now suppose that $\{\alpha_k\}$ is any sequence such that (2.4) holds. Then the preceding formulas hold for arbitrarily large values of $p$. By (3.2) with $k=p+1$ we have, remembering that $f_{p+1}(w;t)=t$:

$$(3.13)\qquad h_{p+1}(w;t)=\frac{1-\beta_{p+1}t}{1+w\beta_{p+1}t}.$$

If we take $t=t_{p+1}=1/\beta_{p+1}$, then $|t_{p+1}|=1$ and $h_{p+1}(w;t_{p+1})\equiv 0$. Then, by (3.10) and (3.11),

$$\frac{1+wS_p(w;t_{p+1})}{1-wS_p(w;t_{p+1})}$$

is the $(p+2)$th approximant of the continued fraction (1.2). Hence, by (2.7) and (2.8), the continued fraction converges uniformly for $|w|\leqq r$ for every constant $r$ less than 1, that is, for $z$ in any bounded closed region within $K$, to the function

$$k(w)=\frac{1+wf(w)}{1-wf(w)}=F(z),\qquad z=\frac{4w}{(1-w)^2},$$

in the class $E$. If, on the other hand, (2.3) holds, then we readily verify that

$$\frac{1+wS_n(w;\alpha_{n+1})}{1-wS_n(w;\alpha_{n+1})}=F(z),$$

a function in the class $E$, is equal to the $(n+2)$th approximant of the continued fraction (1.2) if $1-\alpha_{n+1}\beta_{n+1}=0$, and is equal to the $(n+3)$th approximant, with $g_{n+2}=1$, if $1-\alpha_{n+1}\beta_{n+1}\neq 0$.

Conversely, if we start with a function $F(z)$ in the class $E$, we see immediately that there is determined uniquely a continued fraction (1.2) or a terminating continued fraction of this form, whose value is $F(z)$.

The proof of the theorem stated in the introduction is now complete. We remark that the condition (b) of (1.1) may be easily removed. If we do this, the result may then be stated as follows.

THEOREM. *If* $c>0$, $0<g_p<1$, $-\infty<r_{p-1}<+\infty$, $p=1, 2, 3, \cdots$, *then the continued fraction*

$$\cfrac{c(1+z)^{1/2}}{1+ir_0(1+z)^{1/2}+\cfrac{g_1z}{1+ir_1(1+z)^{1/2}+\cfrac{(1-g_1)g_2z}{1+ir_2(1+z)^{1/2}+\cfrac{(1-g_2)g_3z}{1+ir_3(1+z)^{1/2}+\cdots}}}} \tag{3.14}$$

*converges uniformly over every bounded closed region in the domain* $K$, *and its value is a function* $F(z)$ *which is analytic and has a positive real part throughout* $K$. *Conversely, if* $F(z)$ *is any function with these properties, then there is a uniquely determined continued fraction of the form* (3.14), *or a terminating continued fraction of this form in which the last* $g_p$ *which appears may be unity, whose value is* $F(z)$.

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