A NOTE ON THE DU BOIS-REYMOND EQUATIONS IN THE CALCULUS OF VARIATIONS

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1. Introduction. McShane [2][1] and Tonelli [4] have given conditions on the integrand function which insure that every absolutely continuous solution of the problem of minimizing the integral

$$(1.1) \qquad\qquad I[y] = \int_{x_1}^{x_2} f(x, y, y')dx$$

satisfies the du Bois-Reymond form of the Euler equations. The principal purpose of the present note is to give an alternate proof of these results of McShane and Tonelli. In addition to being simpler in detail than the previous proofs, the differentiability theorems of §§2 and 3 involve weaker hypotheses than the corresponding theorems of McShane and Tonelli, both in regard to conditions on the integrand function and in regard to the class of arcs considered. Basically, the present proof is intimately related to the proof of the fundamental lemma as given by Bliss [1, pp. 20–21].

2. A general differentiability theorem. Suppose that for (x, y, r) $= (x, y_1, \cdots, y_n, r_1, \cdots, r_n)$ in a region R consisting of all values (x, y, r) satisfying $x_1 \leqq x \leqq x_2$, y in an open region Δ of $(y_1, \cdots, y_n)$-space, and $r = (r_1, \cdots, r_n)$ arbitrary, the integrand function $f(x, y, r)$ satisfies the following conditions:

(H$_1$) *For fixed values of (y, r), $f(x, y, r)$ is finite and measurable on* x_1x_2;

(H$_2$) *For fixed values of x, $f(x, y, r)$ is of class C' in (y, r).*

We shall be interested in proving a differentiability property of an arc $y_i = y_i(x)$ which minimizes $I[y]$ in a class of arcs K. Unquestionably, the case in which K consists of absolutely continuous arcs is the most important. Our argument, however, is not complicated by allowing K to be any prescribed class of arcs which possesses the following properties:

(P$_0$) *If the arc $y_i = y_i(x)$ $(x_1 \leqq x \leqq x_2)$ belongs to K, then for each x on x_1x_2 the point $y = [y_i(x)]$ is in the region Δ;*

(P$_1$) *If $y_i = y_i(x)$ $(x_1 \leqq x \leqq x_2)$ belongs to K, then the functions $y_i(x)$ are continuous on x_1x_2 and the derivatives $y_i'(x)$ exist and are finite a. e.*

[1] Numbers in brackets refer to the Bibliography at the end of the paper.

(almost everywhere) on this interval;

(P₂) *If* $y_i = y_i(x)$ $(x_1 \leq x \leq x_2)$ *belongs to* K, *and the functions* $\eta_i(x)$ *are Lipschitzian on* $x_1 x_2$ *and satisfy* $\eta_i(x_1) = 0 = \eta_i(x_2)$, *then for a suitably small in absolute value the arc* $y_i = y_i(x) + a\eta_i(x)$ $(x_1 \leq x \leq x_2)$ *also belongs to* K.

Actually, in the following proof we use (P₂) only for sets of functions $\eta_i(x)$ for which $n - 1$ of these functions are identically zero on $x_1 x_2$.

Along an arc of K the function $f(x, y(x), y'(x))$ is defined a.e. on $x_1 x_2$. In considering (1.1), and similar integrals, it is to be understood that the integrand is the indicated function where this function is well defined, and equal to zero elsewhere on $x_1 x_2$. Correspondingly, a function $F(x)$ which is defined a.e. on $x_1 x_2$ will be said to be measurable on $x_1 x_2$ if the function which is equal to $F(x)$ at the points where it is defined, and to zero elsewhere on $x_1 x_2$, is measurable on this interval. In view of (H₁) and (H₂), and the measurability of a function that is the limit a.e. of a sequence of measurable functions [3, p. 122], it follows that along an arc y of K the function f and its partial derivatives with respect to y_j and r_j $(j = 1, \cdots, n)$ are measurable on $x_1 x_2$.

It is to be noted that we do not suppose that the integral (1.1) exists and is finite for every arc of K. An arc y of K is said to minimize (1.1) in this class if $I[y]$ is finite, and for arbitrary arcs Y of K for which $I[Y]$ is finite we have $I[Y] \geq I[y]$.

The following lemma will be used in the proof of the subsequent theorems.

LEMMA 2.1. *Let* Δ^* *be a given bounded and closed subregion of* Δ *in* y-*space, and for a given positive integer* N *denote by* R_N *the subregion of* R *consisting of all* (x, y, r) *satisfying* $x_1 \leq x \leq x_2$, y *in* Δ^*, *and* $|r_i| \leq N$ $(i = 1, 2, \cdots, n)$. *Then there exists a corresponding function* $\psi_N(x)$ *which is finite and measurable on* $x_1 x_2$, *and such that for* (x, y, r) *in* R_N *we have*

(2.1)
$$|f(x, y, r)| \leq \psi_N(x), \qquad |f_{y_i}(x, y, r)| \leq \psi_N(x),$$
$$|f_{r_i}(x, y, r)| \leq \psi_N(x) \qquad (i = 1, 2, \cdots, n).$$

In view of hypotheses (H₁) and (H₂) it follows that for fixed values of (y, r) the partial derivatives of f with respect to y_i and r_i $(i = 1, \cdots, n)$ are finite and measurable on $x_1 x_2$. Consequently, in R the function

(2.2) $g(x, y, r) = |f(x, y, r)| + \sum_{i=1}^{n} [|f_{y_i}(x, y, r)| + |f_{r_i}(x, y, r)|]$

is continuous in (y, r) for fixed values of x on x_1x_2, and measurable on x_1x_2 for fixed values of (y, r). Now R_N is the product of the interval $x_1 \leq x \leq x_2$ and the closed and bounded region D_N in (y, r) space consisting of all sets (y, r) having y in Δ^* and $|r_i| \leq N$ $(i=1, 2, \cdots, n)$. For each x on x_1x_2 define the functional value $\psi_N(x)$ as the maximum of $g(x, y, r)$ on D_N. In view of the continuity of $g(x, y, r)$ as a function of (y, r) it follows that $\psi_N(x)$ is finite on x_1x_2. Moreover, $\psi_N(x)$ is measurable on this interval [3, p. 122], since it is the limit superior of a sequence of measurable functions $g(x, y^{(k)}, r^{(k)})$, where $(y^{(k)}, r^{(k)})$ $(k=1, 2, \cdots)$ is a countable sequence dense in D_N. The function $\psi_N(x)$ thus defined clearly satisfies the conclusion of the lemma.

THEOREM 2.1. *Suppose that* $y_i = y_i(x)$ $(x_1 \leq x \leq x_2)$ *is a minimizing arc for* (1.1) *in a class* K, *and, in addition to hypotheses* (H_1) *and* (H_2), *the integrand satisfies with this arc the further condition*:

$(H_{3;j})$ *There exists a constant* $\delta > 0$ *and a function* $\phi_j(x)$ *integrable on* x_1x_2 *and such that for* $|\bar{y}_j - y_j(x)| < \delta$, $\bar{y}_i = y_i(x)$ $(i \neq j)$, *we have*

$$|f_{y_j}(x, \bar{y}, y'(x))| \leq \phi_j(x)$$

a.e. on x_1x_2. *Then there exists a corresponding constant* c_j *such that*

$$z_j(x) \equiv f_{r_j}(x, y(x), y'(x)) - \int_{x_1}^{x} f_{y_j}(t, y(t), y'(t))dt$$

is equal to c_j *a.e. on* x_1x_2.

Condition $(H_{3;j})$ implies the existence of the integral appearing in the definition of $z_j(x)$; it also implies that the integral

$$\int_{x_1}^{x_2} f(x, Y(x), y'(x))dx$$

exists for all arcs $Y_i = Y_i(x)$ of K satisfying $|Y_j(x) - y_j(x)| < \delta$, $Y_i(x) = y_i(x)$ $(i \neq j)$ on x_1x_2, since for such arcs,

$$(2.3) \qquad |f(x, Y(x), y'(x))| \leq |f(x, y(x), y'(x))| + \delta\phi_j(x).$$

Corresponding to a positive integer N, let E_N denote the set of all points x on x_1x_2 at which the derivatives y_i' exist and $|y_i'| \leq N-1$ $(i=1, \cdots, n)$. We may suppose that N is sufficiently large to insure that $m(E_N)$, the measure of E_N, is positive, since E_N is a subset of E_{N+1} $(N=1, 2, \cdots)$ and $\lim_{N \to \infty} m(E_N) = x_2 - x_1$. Furthermore, let the region Δ^* of Lemma 2.1 be such that it contains in its interior all the points $y = [y_i(x)]$ $(x_1 \leq x \leq x_2)$ belonging to the supposed minimizing arc, and denote by $E_{N;M}$ the subset of E_N where the function $\psi_N(x)$

of that lemma satisfies the condition $\psi_N(x) \leqq M$ $(M = 1, 2, \cdots)$. Since $E_{N;M}$ is a subset of $E_{N;M+1}$ and $\lim_{M \to \infty} m(E_{N;M}) = m(E_N) > 0$, we may also assume that M is so chosen that $m(E_{N;M}) > 0$.

Now on $E_{N;M}$ the function $z_j(x)$ defined in $(H_{3;j})$ is bounded and measurable, hence integrable. Consequently, there is a unique constant c_j such that

$$(2.4) \qquad 0 = \int_{E_{N;M}} [z_j(x) - c_j] dx = \int_{E_{N;M}} z_j(x) dx - c_j m(E_{N;M}).$$

Now define $\eta_i(x) \equiv 0$ for $i \neq j$, and

$$(2.5) \qquad \eta_j(x) = \int_{x_1}^{x} [z_j(t) - c_j] I_{N;M} dt,$$

where $I_{N;M}$ is the characteristic function of the set $E_{N;M}$. The functions $\eta_i(x)$ are clearly Lipschitzian on $x_1 x_2$ and satisfy $\eta_i(x_1) = 0 = \eta_i(x_2)$; moreover, $\eta_j' = 0$ a.e. on $CE_{N;M}$, the complement of $E_{N;M}$ on $x_1 x_2$. In view of the Lipschitzian character of the functions $\eta_i(x)$ and property (P_2), there exists a positive constant a_0 such that for $|a| < a_0$ the arc $y_i = y_i(x) + a \eta_i(x)$ belongs to K and lies in the region Δ^* of Lemma 2.1. We shall also suppose that a_0 is chosen so small that $a_0 |\eta_j(x)| < \delta$ on $x_1 x_2$ and $a_0 |\eta_j'(x)| \leqq 1$ a.e. on this interval. In particular, we have that $|y_j'(x) + a \eta_j'(x)| \leqq N$ a.e. on E_N, and hence a.e. on the subset $E_{N;M}$. For such values of a the integral $I[y + a\eta]$ exists and is finite, and

$$(2.6) \qquad \begin{aligned} I[y + a\eta] &= \int_{CE_{N;M}} f(x, y + a\eta, y') dx \\ &\quad + \int_{E_{N;M}} f(x, y + a\eta, y' + a\eta') dx. \end{aligned}$$

The existence and finiteness of the first integral of (2.6) follows from (2.3) for $Y(x) = y(x) + a\eta(x)$. The existence and finiteness of the second integral of (2.6) follows from the measurability of the integrand, together with the fact that a.e. on $E_{N;M}$ the set $(x, y(x) + a\eta(x), y'(x) + a\eta'(x))$ is in the region R_N of Lemma 2.1, and consequently $|f(x, y(x) + a\eta(x), y'(x) + a\eta'(x))| \leqq \psi_N(x) \leqq M$ on this set.

Now, in view of $(H_{3;j})$, for $|a| < a_0$ the partial derivative of $f(x, y(x) + a\eta(x), y'(x))$ with respect to a exists and does not exceed in absolute value the integrable function $|\eta_j(x)| \phi_j(x)$, which, in turn, is not greater than $(\delta/a_0) \phi_j(x)$. Moreover, a.e. on $E_{N;M}$ the partial derivative of $f(x, y(x) + a\eta(x), y'(x) + a\eta'(x))$ with respect to a exists

and does not exceed in absolute value $(|\eta_i(x)| + |\eta_i'(x)|)\psi_N(x)$
$\leq [(\delta+1)/a_0]M$. It then follows [3, p. 216] that the derivative of
(2.6) with respect to a exists for $|a| < a_0$, and is given by the usual
differentiation under the integral sign. For $a = 0$ this derivative is

$$
\begin{aligned}
&\int_{CE_{N;M}} f_{v_j}\eta_j\,dx + \int_{E_{N;M}} [f_{v_j}\eta_j + f_{r_j}\eta_j']\,dx \\
(2.7) \\
&\qquad = \int_{x_1}^{x_2} [f_{v_j}\eta_j + f_{r_j}\eta_j']\,dx = \int_{x_1}^{x_2} z_j\eta_j'\,dx,
\end{aligned}
$$

where the arguments of the partial derivatives of f occurring in (2.7)
are $(x, y(x), y'(x))$. The last relation follows by the usual integration
by parts, and use of the conditions $\eta_j(x_1) = 0 = \eta_j(x_2)$. Since the mini-
mizing property of y requires that this derivative be zero, and as the
integral of $\eta_i'(x)$ over x_1x_2 is equal to zero, it follows, in particular,
that

$$
\begin{aligned}
0 &= \int_x^{x_2} [z_j(x) - c_j]\eta_j'(x)\,dx = \int_{x_1}^{x_2} [z_j(x) - c_j]^2 I_{N;M}\,dx \\
(2.8) \\
&= \int_{E_{N;M}} [z_j(x) - c_j]^2\,dx.
\end{aligned}
$$

From (2.8) we have that $z_j(x) = c_j$ a.e. on $E_{N;M}$. Now the constant
c_j is seemingly dependent on N and M. Since, however, $E_{N;M}$ is a
subset of $E_{N;M+1}$ $(M = 1, 2, \cdots)$ and $\lim_{M\to\infty} m(E_{N;M}) = m(E_N)$, it is
seen that a.e. on E_N we have $z_j(x) = c_j$, where c_j is the constant de-
termined by any set $E_{N;M}$ for which $m(E_{N;M}) > 0$. Similarly, since E_N
is a subset of E_{N+1} $(N = 1, 2, \cdots)$ and $\lim_{N\to\infty} m(E_N) = x_2 - x_1$, it fol-
lows that $z_j(x) = c_j$ a.e. on x_1x_2.

3. **A second differentiability theorem.** We shall now prove another
differentiability property of a minimizing arc for (1.1) under the fur-
ther hypotheses:

(H) $f(x, y, r)$ *is of class* C' *in all its arguments on* R;

(P_2') *If* $y_i = y_i(x)$ $(x_1 \leq x \leq x_2)$ *is an arc of* K, *and the function* $\zeta(x)$
is such that $\zeta(x_1) = x_1$, $\zeta(x_2) = x_2$, *while there is a constant* $k \geq 1$ *such that*
$(x' - x'')/k \leq \zeta(x') - \zeta(x'') \leq k(x' - x'')$ *for arbitrary values* x', x'' *satis-*
fying $x_1 \leq x'' < x' \leq x_2$, *then* $y_i = y_i(\zeta(x))$ *is also an arc of* K.

Condition (P_2') is satisfied if K is either the class of arcs of bounded
variation, the class of absolutely continuous arcs, or the class of Lip-
chitzian arcs.

THEOREM 3.1. *Suppose that* $y_i = y_i(x)$ $(x_1 \leq x \leq x_2)$ *is a minimizing*

arc for (1.1) *in a class* K *satisfying* (P_0), (P_1) *and* (P_2'), *and, in addition to hypothesis* (H), *the integrand satisfies with this arc the further condition*:

(H$_{3;0}$) *There exists a constant* δ *and a function* $\phi_0(x)$ *integrable on* x_1x_2 *and such that if* x *and* $\bar{x}$ *are on this interval and satisfy* $|\bar{x}-x| < \delta$, *then a.e. on* x_1x_2,

$$| f_x(\bar{x}, y(x), y'(x)) | \leqq \phi_0(x).$$

Then there exists a constant c_0 *such that*

$$z(x) \equiv f(x, y(x), y'(x)) - \sum_{i=1}^{n} y_i'(x)f_{r_i}(x, y(x), y'(x))$$

$$- \int_{x_1}^{x} f_x(t, y(t), y'(t))dt$$

is equal to c_0 *a.e. on* x_1x_2.

Condition (H$_{3;0}$) implies, in particular, that the partial derivative $f_x(x, y(x), y'(x))$ is integrable on x_1x_2. Corresponding to a positive integer N, again let E_N denote the subset of x_1x_2 on which the derivatives $y_i'(x)$ exist and satisfy $|y_i'(x)| \leqq N-1$ $(i=1, \cdots, n)$. Hypothesis (H) implies that along the minimizing arc the integrand f and its partial derivatives with respect to r_i $(i=1, \cdots, n)$ are bounded functions of x on E_N. As in the proof of Theorem 2.1, we may suppose that N is so large that $m(E_N) > 0$. Now on E_N the function $z(x)$ is bounded and measurable, and hence integrable. Let c_0 be the constant determined by the condition

$$0 = \int_{E_N} [z(x) - c_0]dx = \int_{E_N} z(x)dx - c_0m(E_N),$$

and set

$$\eta(x) = \int_{x_1}^{x} [z(t) - c_0]I_N dt,$$

where I_N is the characteristic function of the set E_N. Then $\eta(x)$ is Lipschitzian on x_1x_2, $\eta(x_1) = 0 = \eta(x_2)$, and for a_0 sufficiently small the function $T(t; a) = t + a\eta(t)$ is such that $(t' - t'')/2 \leqq T(t'; a) - T(t''; a) \leqq 2(t' - t'')$ for all values of a such that $|a| < a_0$, and for arbitrary t', t'' satisfying $x_1 \leqq t'' < t' \leqq x_2$; moreover, $T(x_1; a) = x_1$, $T(x_2; a) = x_2$. Hence for such values of a the equation

$$x = t + a\eta(t) \equiv T(t; a)$$

has a unique solution $t = \zeta(x; a)$, which is Lipschitzian on x_1x_2 and

such that $(x'-x'')/2 \leqq \zeta(x'; a) - \zeta(x''; a) \leqq 2(x'-x'')$ for arbitrary values x', x'' satisfying $x_1 \leqq x'' < x' \leqq x_2$. We have also that $\zeta(x_1, a)$ $= x_1$, $\zeta(x_2, a) = x_2$, while $\zeta(x; 0) \equiv x$ on $x_1 x_2$. It then follows from (P$_2'$) that $y_i(x; a) = y_i(\zeta(x; a))$ $(x_1 \leqq x \leqq x_2)$ defines an arc of K such that $y_i(x; 0) = y_i(x)$.

From known theorems on the change of variable in a simple integral [3, p. 215] we then have that the integral (1.1) calculated along the arc $y_i = y_i(x; a)$ is finite and equal to

$$\int_{x_1}^{x_2} f(t + a\eta(t), y(t), y'(t)/[1 + a\eta'(t)]) [1 + a\eta'(t)] dt$$

(3.1)
$$= \int_{E_N} f(t + a\eta(t), y(t), y'(t)/[1 + a\eta'(t)]) [1 + a\eta'(t)] dt$$

$$+ \int_{CE_N} f(t + a\eta(t), y(t), y'(t)) dt.$$

Since $|y_i'| \leqq N-1$ on E_N, it follows from (H) that a.e. on E_N the partial derivative with respect to a of the integrand of the first integral of (3.1) exists for $|a| < a_0$; moreover, this partial derivative is uniformly bounded for x on this subset of E_N and $|a| < a_0$. Hypotheses (H) and (H$_{8;0}$) imply that for sufficiently small values of a the partial derivative with respect to a of the integrand of the second integral exists and does not exceed in absolute value the integrable function $|\eta(t)| \phi_0(t)$. It then follows from the previously applied theorem on differentiability of integrals that the derivative of (3.1) exists for $a = 0$, and is given by

(3.2)
$$\int_{x_1}^{x_2} \left[\left(f - \sum_{i=1}^{n} y_i' f_{r_i} \right) \eta' + f_x \eta \right] dx,$$

where the arguments of f and its partial derivative occurring in (3.2) are $(x, y(x), y'(x))$.

In view of the minimizing property of the arc $y_i = y_i(x)$, the integrability of f_x along this arc, and the conditions $\eta(x_1) = 0 = \eta(x_2)$, we conclude that

$$0 = \int_{x_1}^{x_2} z(x)\eta'(x)dx = \int_{x_1}^{x_2} [z(x) - c_0]\eta'(x)dx$$

$$= \int_{x_1}^{x_2} [z(x) - c_0]^2 I_N dx = \int_{E_N} [z(x) - c_0]^2 dx.$$

That is, $z(x) = c_0$ a.e. on E_N. As E_N is a subset of E_{N+1} $(N = 1, 2, \cdots)$

and $\lim_{N\to\infty} m(E_N) = x_2 - x_1$, it then follows that $z(x) = c_0$ a.e. on $x_1 x_2$.

4. **Further remarks.** The general differentiability theorems of §§2 and 3 involve, respectively, hypotheses $(H_{3;j})$ and $(H_{3;0})$ for the considered minimizing arc. Now suppose that the integrand function $f(x, y, r)$ satisfies (H_1) and (H_2), and, in addition to (P_0), (P_1) and (P_2), the class K contains only arcs $y_i = y_i(x)$ such that the derivative functions $y_i'(x)$ are integrable on $x_1 x_2$. Then every arc of K for which $I[y]$ is finite satisfies $(H_{3;j})$ whenever the integrand function satisfies the following condition:

$(H_{3;j}^*)$ *There exist positive constants M_1, M_2, δ such that if (x, y, r) and $(x, \bar{y}, r)$ are points of R with $|\bar{y}_j - y_j| < \delta$, $\bar{y}_i = y_i$ ($i \neq j$) then*

$$(4.1) \quad |f_{y_j}(x, \bar{y}, r)| \leq M_1 |f(x, y, r)| + M_2 \left[1 + \sum_i r_i^2 \right]^{1/2}.$$

Correspondingly, if $f(x, y, r)$ satisfies (H), and, in addition to (P_0), (P_1) and (P_2'), the class K contains only arcs $y_i = y_i(x)$ such that the functions $y_i'(x)$ are integrable on $x_1 x_2$, then every arc of K for which $I[y]$ is finite satisfies $(H_{3;0})$ under the following additional hypothesis:

$(H_{3;0}^*)$ *There exist positive constants M_1, M_2, δ such that if (x, y, r) and $(\bar{x}, y, r)$ are points of R with $|\bar{x} - x| < \delta$, then*

$$(4.2) \quad |f_x(\bar{x}, y, r)| \leq M_1 |f(x, y, r)| + M_2 \left[1 + \sum_i r_i^2 \right]^{1/2}.$$

The above hypotheses $(H_{3;j}^*)$ and $(H_{3;0}^*)$ have been used by McShane [2]. In connection with the above remarks it is to be noted that if the term $M_2[1 + \sum_i r_i^2]^{1/2}$ in (4.1) and (4.2) is replaced by M_2, then the condition that the functions $y_i'(x)$ be integrable on $x_1 x_2$ is not needed to insure that every arc of K for which $I[y]$ is finite will satisfy the corresponding condition $(H_{3;j})$ or $(H_{3;0})$.

Finally, it is to be remarked that as a very special case of Theorem 2.1 one has the following generalization of the fundamental lemma of the calculus of variations:

If $M(x)$ is finite and measurable on $x_1 x_2$, and the integral

$$\int_{x_1}^{x_2} M(x) \eta'(x) dx$$

is zero for every Lipschitzian function $\eta(x)$ satisfying $\eta(x_1) = 0 = \eta(x_2)$, and for which the above integral exists and is finite, then there exists a constant c such that $M(x) = c$ a.e. on $x_1 x_2$.

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