$\rho(r)=\xi(r)$ for $t_2\leqq r\leqq s_2$ where t_2 $(<s_2)$ is the point nearest to s_2 at which $\xi(t_2)=\sigma(t_2)$. If $\xi(s_2)=\sigma(s_2)$, then let $t_2=s_2$. For $r<t_2$ let $\rho(r) =\rho(t_2)+\log\log\log t_2-\log\log\log r$ for $u_1\leqq r\leqq t_2$ where u_1 $(<t_2)$ is the point of intersection of $y=\rho$ with $y=\rho(t_2)+\log\log\log t_2-\log\log\log r$.

Let $\rho(r)=\rho$ for $r_1\leqq r\leqq u_1$. It is always possible to choose r_2 so large that $r_1<u_1$. We repeat the procedure and note that

$$\rho(r) \geqq \xi(r) \geqq \sigma(r)$$

and $\rho(r)=\sigma(r)$ for $r=t_1, t_2, t_3, \cdots$. Hence $\lim_{r\to\infty}\rho(r)=\rho$, and

$$\limsup_{r\to\infty}\frac{\log M(r)}{r^{\rho(r)}} = 1.$$

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A NOTE ON THE SPECTRAL THEOREM

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1. Introduction. Although the connections between the spectral resolution of a self-adjoint transformation in Hilbert space, the moment problem, and Riesz' integral representation [1][1] for linear functionals on the space C are known (cf. Stone [2], Murray [3], Widder [4], Lengyel [5]), the following elementary derivation of the spectral theorem from the Riesz theorem exhibits the connections in, perhaps, the simplest light. We consider only *bounded* self-adjoint transformations H; one can treat an unbounded H by considering $(I+H^2)^{-1}$, which is bounded and self-adjoint [3, p. 95]. Note that the derivation does not involve the separability of the Hilbert space $\mathfrak{H}$.

2. Six lemmas. Let H be a self-adjoint transformation with the bounds a, b—that is, $a\|f\|^2\leqq(Hf, f)\leqq b\|f\|^2$ for all $f\in H$, and $\|H\| =\max(|a|, |b|)$. Denote by C the space of continuous real-valued functions defined on the closed interval (a, b), with $\|f(x)\|=\max|f(x)|$ $(a\leqq x\leqq b)$. Let $p(x)=\sum_0^n c_ix^i$ be any polynomial with real coefficients, and let $p(H)$ be the corresponding transformation $p(H)=\sum_0^n c_iH^i$, where $H^0=I$.

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[1] Numbers in brackets refer to the references cited at the end of the paper.

LEMMA 1. $\|p(H)\| \leqq \|p(x)\| = \max_{x \in (a,b)} |p(x)|$.

Murray [3, p. 82] gives a proof of the lemma; we remark only that the general case can be reduced to the case of a self-adjoint H in n-dimensional Euclidean space. The inequality is almost obvious when H is represented by a finite diagonal matrix; and the lemma is essentially equivalent to the statement that every n^2-symmetric matrix can be reduced to diagonal form by unitary transformations.

Given any two elements $f, g \in \mathfrak{H}$, consider the expression $(p(H)f, g)$, taking on real or complex values, as an operator defined over the linear set of polynomials included in C. The linearity of the operator is obvious; its continuity follows from Lemma 1:

$$|(p(H)f, g)| \leqq \|p(H)\| \cdot \|f\| \cdot \|g\| \leqq \{\|f\| \cdot \|g\|\} \|p(x)\|.$$

Since the polynomials are dense in C (Weierstrass approximation theorem), the operator can be extended uniquely to a linear functional defined over all C, without increase of norm. The Riesz theorem [1] then yields immediately:

LEMMA 2. *There exists a function of bounded variation* $\rho(\lambda; f, g)$ $(a \leqq \lambda \leqq b)$ *such that*

$$(p(H)f, g) = \int_a^b p(\lambda) d\rho(\lambda; f, g),$$

the uniqueness of ρ *being assured by the normalization conditions*: $\rho(a; f, g) = 0$ *and* $\rho(\lambda; f, g) = \rho(\lambda+0; f, g)$, $a < \lambda < b$. $\rho(\lambda; f, f)$ *is real, and* $\mathrm{Var}_{(a,b)} \rho(\lambda; f, f) \leqq \|f\|^2$.

The structure of the $\rho(\lambda; f, g)$ is implied by the following elementary properties of the Stieltjes integral (cf. Widder [4]).

LEMMA 3. *Let* $f(x)$, $g(x)$ *be continuous and* $\gamma(x)$ *be a normalized function of bounded variation on* (a, b). *Then* $G(x) = \int_a^x g(t) d\gamma(t)$ *is a normalized function of bounded variation, and* $\int_a^b f(t) dG(t) = \int_a^b f(t) g(t) d\gamma(t)$. *If* $\int_a^b t^n d\gamma(t) = 0$ $(n = 0, 1, 2, \cdots)$, *then* $\gamma(t) \equiv 0$.

LEMMA 4. (1) $\rho(b; f, g) = (f, g)$, $0 \leqq \rho(\lambda; f, f) \leqq \|f\|^2$;

(2) $\rho(\lambda; f_1 + f_2, g) = \rho(\lambda; f_1, g) + \rho(\lambda; f_2, g)$;

(3) $\rho(\lambda; cf, g) = c\rho(\lambda; f, g)$;

(4) $\rho(\lambda; f, g) = \bar{\rho}(\lambda; g, f)$.

The equality in (1) arises on setting $p(\lambda) = 1$ in Lemma 2. But then $\rho(\lambda; f, f)$ is nondecreasing, whence the inequality. The proof of (2) is typical of the remaining statements:

$$\int_a^b \lambda^n d\rho(\lambda; f_1 + f_2, g) = (H^n(f_1 + f_2), g) = (H^n f_1, g) + (H^n f_2, g)$$
$$= \int_a^b \lambda^n d\rho(\lambda; f_1, g) + \int_a^b \lambda^n d\rho(\lambda; f_2, g)$$
$$= \int_a^b \lambda^n d\{\rho(\lambda; f_1, g) + \rho(\lambda; f_2, g)\}.$$

Set $\gamma(\lambda) = \rho(\lambda; f_1 + f_2, g) - \{\rho(\lambda; f_1, g) + \rho(\lambda; f_2, g)\}$ and apply Lemma 3.

Lemma 4 and the representation theorem [2, p. 63] for bounded, bilinear symmetric functionals defined over a Hilbert (or unitary) space yield the following theorem:

LEMMA 5. *There exists a set of bounded self-adjoint transformations* $F(\lambda)$ $(a \leqq \lambda \leqq b)$ *such that* $(F(\lambda)f, g) = \rho(\lambda; f, g)$. $F(a) = 0$, *and* $F(b) = I$.

That the $F(\lambda)$ are projections is a consequence of the following result:

LEMMA 6. $F(\mu)F(\lambda) = F(\nu)$, *where* $\nu = \min(\mu, \lambda)$. *In particular,* $F^2(\lambda) = F(\lambda)$.

For, let m and n be arbitrary $(m, n = 0, 1, 2, \cdots)$. Then

$$\int_a^b \lambda^n d\left\{\int_a^\lambda \mu^m d(F(\mu)f, g)\right\} = \int_a^b \lambda^n \cdot \lambda^m d(F(\lambda)f, g) = (H^{n+m}f, g)$$
$$= (H^n f, H^m g) = \int_a^b \lambda^n d(F(\lambda)f, H^m g)$$
$$= \int_a^b \lambda^n d(H^m F(\lambda)f, g)$$
$$= \int_a^b \lambda^n d\left\{\int_a^b \mu^m d(F(\mu)F(\lambda)f, g)\right\}.$$

Application of Lemma 3 yields

$$\int_a^b \mu^m d(F(\mu)F(\lambda)f, g) = \int_a^\lambda \mu^m d(F(\mu)f, g) = \int_a^b \mu^m d(F(\nu)f, g);$$

and another application, $F(\mu)F(\lambda) = F(\nu)$, where $\nu = \min(\mu, \lambda)$.

3. The spectral theorem. In order to put our results into standard form we modify the definition of the family of projections by setting

$E(\lambda)=0$ $(\lambda<a)$, $E(\lambda)=F(\lambda+0)$ $(a\leqq\lambda<b)$, $E(\lambda)=I$ $(\lambda\geqq b)$. The $E(\lambda)$ thus obtained form a *finite resolution of the identity*: They are all projections; $E(\lambda)E(\mu)=E(\nu)$, where $\nu=\min(\mu, \lambda)$; $E(\lambda+0)=E(\lambda)$ $(-\infty<\lambda<\infty)$; $(E(\lambda)f, g)$ is of bounded variation; $E(\lambda)=0$ $(\lambda<a)$ and $E(\lambda)=I$ $(\lambda\geqq b)$.

To allow for the change in definition at the point $\lambda=a$ we modify the limits of integration from (a, b) to $(a-\epsilon, b)$, ϵ being an arbitrary positive number. Our final result is then the following:

SPECTRAL THEOREM. *Let H be a bounded self-adjoint transformation with bounds a, b. Then there exists a finite resolution of the identity $E(\lambda)$ such that*

$$(p(H)f, g) = \int_{a-\epsilon}^{b} p(\lambda)d(E(\lambda)f, g),$$

ϵ being an arbitrary positive number.

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