247. A. C. Sugar: *An elementary exposition of the relaxation method.*

This is, as labeled, an elementary exposition. The purpose of this paper is to exhibit the simplicity and power of the relaxation method, to complete and explain previous sketchy or obscure discussions of this subject. It is also intended to direct attention to some mathematical, physical, and philosophical questions that may be raised in connection with this method. (Received May 29, 1946.)

248. A. C. Sugar: *On the relaxation-matrix method of solving boundary value problems.*

This is a continuation of the study of the use of relaxation and iterative methods of inverting the matrices of the systems of equations obtained from boundary value problems by finite difference methods. In a previous paper, entitled *The use of invariant inverted matrices for the approximate solution of classes of boundary value problems*, the writer inverted matrices by relaxation methods and applied them to the solution of simple illustrative problems. In the present paper, this work is continued and applications are made to some of the typical boundary value problems of mechanics. (Received May 29, 1946.)

249. A. C. Sugar: *The use of invariant inverted matrices for the simultaneous approximate solution of classes of boundary value problems.* Preliminary report.

The writer considers the simultaneous approximate solution of classes of boundary value problems. Each class consists of the totality of boundary value problems having the same differential equation and the same boundary but different boundary conditions. Using finite difference methods it is shown that the derived system of equations will have an inverse matrix M, invariant over the class, which may be determined by relaxation methods. A solution of any member of the class may then be obtained by multiplying M by a column matrix defined by the corresponding boundary values. This paper will be primarily concerned with the application of this method to Laplace's equation. The effect of modifications of the boundary on M will be considered. This technique may be applied to many other types of differential equations. This is true, in particular, of Poisson's equation and of nonlinear equations containing the Laplace operator, since, as far as the algebraic treatment is concerned, these two types may be treated as Laplace equations with altered boundary conditions. Finally, the possibility of considering M as an approximation or an analogue of Green's function is studied. (Received April 16, 1946.)

Geometry

250. L. M. Blumenthal: *Superposability in elliptic spaces.*

Two subsets of a metric space M are *superposable* provided a congruent (that is, one-to-one, distance-preserving) mapping of M onto itself exists which maps one subset onto the other. In spaces most studied (euclidean, spherical, and so on) congruence of subsets implies that they are superposable, but this is not the case in elliptic spaces $E_{n,r}$ for $n>1$, and hence this property cannot be expressed in metric terms alone. This paper shows that congruent but not superposable subsets of $E_{n,r}$ fall into two classes (a) congruent subsets contained irreducibly in different dimensional subspaces and (b) those contained irreducibly in subspaces of the same dimension. By means of attaching to each elliptic m-tuple a class of equivalent matrices, necessary and sufficient conditions for superposability of elliptic subsets are obtained in terms of equivalence of the corresponding classes of attached matrices. These conditions are applied to yield new properties of elliptic spaces and quadratic forms. (Received April 16, 1946.)

251. Herbert Busemann: *Intrinsic area.*

Recent examples of Besicovitch have turned the attention to the geometric concept of area. Although the requirement that area should be intrinsic (that is, depend only on the geodesic distances on the surface and not on the way the surface is imbedded) is fundamental for all applications in geometry, it has never entered the modern investigations on area. An intrinsic area can be defined for a general class of surfaces in metric spaces in such a way that it has the standard value for euclidean polyhedra and smooth surfaces in Riemann spaces. Moreover the definition attributes an area to surfaces in Finsler spaces. This value is for smooth surfaces the only area which coincides with the customary area for elementary surfaces, and for which the surface with the greater intrinsic distances has always the greater area. The paper will appear in the Annals of Mathematics. (Received May 27, 1946.)

252. N. A. Court: *On the biratio of the altitudes of a tetrahedron.*

The three pairs of opposite edges a, a'; b, b'; c, c' of a tetrahedron (T) are the axes of three orthogonal hyperboloids (aa'), (bb'), (cc') belonging to the same pencil of quadrics which also includes the hyperboloid (H) formed by the altitudes of (T) (Bull. Amer. Math. Soc. vol. 51 (1945) p. 663; Duke Math. J. vol. 13 (1946) pp. 123–128). The biratio (that is, the anharmonic ratio) of the four hyperboloïds considered is equal to the biratio η of the four altitudes of (T). If the three vertices of (T) are fixed, for a given value of η, the fourth vertex lies in a plane perpendicular to the plane formed by the fixed vertices. The biratio of the tetrad of points in which an altitude of (T) meets the faces of the tetrahedron formed by the orthocenters of the faces of (T) is equal to η. (Received May 29, 1946.)

253. Tibor Radó: *The isoperimetric inequality and the Lebesgue definition of surface area.* I.

Let U^* denote the solid unit sphere $u^2+v^2+w^2\leqq 1$, and let $p=T(p^*)$, $p^*\in U^*$, be a topological mapping from U^* into Euclidean three-space. S, D will denote the images, under T, of the boundary and of the interior of U^*, and $|S|$, $|D|$ will denote the three-dimensional measures of S, D. Let A be the Lebesgue area of S, and consider the isoperimetric inequality (*) $V^2\leqq A^3/36\pi$, where V is the volume enclosed by S. In case $|S|>0$, a decision must be made whether the interior volume $V_i=|D|$ or the exterior volume $V_e=|D|+|S|$ is to be used in the inequality (*). An example of Besicovitch shows that (*) generally fails to hold for V_e. In this paper, it is shown that (*) holds in the Besicovitch example for V_i, and further examples, based on an observation of Geöcze made in 1913, are given which indicate that beyond the elementary level the concept of "enclosed" volume must be properly defined if the isoperimetric inequality is to hold. (Received May 27, 1946.)

254. Tibor Radó: *The isoperimetric inequality and the Lebesgue definition of surface area.* II.

Let S^* denote the unit sphere $u^2+v^2+w^2=1$, and let $p=T(p^*)$, $p^*\in S^*$, be a continuous mapping from S^* into Euclidean xyz space. Then T determines a (not necessarily simple) closed surface S. Define an index-function n (x, y, z) as follows: if the point (x, y, z) lies on S, then $n=0$; if (x, y, z) does not lie on S, then n is equal to the topological index of the point (x, y, z) with respect to S. Then $n(x, y, z)$ vanishes outside of a sufficiently large sphere K. Define $V(S)$, the volume enclosed by S, as the integral of $|n(x, y, z)|$ in K if this integral exists, and let $V(S)=\infty$ otherwise. The purpose of the paper is to establish the isoperimetric inequality $V(S)^2 \leqq A(S)^3/36\pi$, where $A(S)$ is the Lebesgue area of S, as a generalization of previous results of Tonelli and Bonnesen. (Received May 29, 1946.)

Statistics and Probability

255. Z. W. Birnbaum: *Tshebysheff inequality for two dimensions.*

For independent random variables X, Y with expectations zero and variances σ_X^2, σ_Y^2 the trivial inequality $P(X^2+Y^2 \geqq T^2) \leqq (\sigma_X^2+\sigma_Y^2)/T^2$ is replaced by a sharp inequality. (Received April 5, 1946.)

256. Mark Kac: *A discussion of the Ehrenfest model.* Preliminary report.

A particle moves along a straight line in steps Δ, the duration of each step being τ. The probabilities that the particle at $k\Delta$ will move to the right or left are $(1/2)(1-k/R)$ and $(1/2)(1+k/R)$ respectively. R and k are integers and $|k| \leqq R$. M. C. Wang and G. E. Uhlenbeck in their paper *On the theory of Brownian motion.* II (Review of Modern Physics vol. 17 (1945) pp. 323–342) discuss this random walk problem and state several unsolved problems. In answer to some of the questions raised the following results are obtained: Let $(1-z)^{R-j}(1+z)^{R+j}=\sum C_k^{(j)} z^k$($j$ an integer), then the probability $P(n, m|s)$ that a particle starting from $n\Delta$ will come to $m\Delta$ after time $t=s\tau$ is equal to $2^{-2R}(-1)^{R+n}\sum(j/R)^s C_{R+j}^{(-n)} C_{R+m}^{(j)}$, where the summation is extended over all j such that $|j| \leqq R$. Also, if R is even the probability $P'(n, 0|s)$ that the particle starting from $n\Delta$ will come to 0 at $t=s\tau$ for the first time is calculated. For $n=0$ this gives a solution of the so-called recurrence time problem first studied on simpler models by Smoluchowski. Through a limiting process in which $\tau\rightarrow 0$, $\Delta\rightarrow 0$, $\Delta^2/2\tau\rightarrow D$, $1/R\tau\rightarrow\beta$, $n\Delta\rightarrow x_0$, $m\Delta\rightarrow x$, $s\tau=t$, one is led to fundamental distributions concerning the velocity of a free Brownian particle. In particular, $P(n, m|s)$ approaches the well known Ornstein-Uhlenbeck distribution. (Received May 23, 1946.)

257. Howard Levene: *A test of randomness in two dimensions.*

A square of side N is divided into N^2 unit cells, and each cell takes on the characteristics A or B with probabilities p and $q=1-p$ respectively, independently of the other cells. A cell is an "upper left corner" if it is A and the cell above and cell to the left are not A. Let V_1 be the total number of upper left corners and let V_2, V_3, V_4 be the number of similarly defined upper right, lower right, and lower left corners respectively. Let $V=(V_1+V_2+V_3+V_4)/4$. It is proved that V is normally distributed in the limit with $E(V)=p(Nq+p)^2$ and $\sigma^2(V)=N^2pq^2(2-10p+22p^2-13p^3)/2$. The conditional limit distribution of V, when p is estimated from the data, and the limit distribution of a related quadratic form are also obtained. These statistics are in a sense a generalization of the run statistics used for testing randomness in one dimension. (Received May 28, 1946.)