tion of linear functions of ordered dyads $X_{m+1}(\alpha)=\sum x_j\alpha_1\alpha_j$ $(j=1, 2, \cdots, m+1)$ with coefficients x_j in a suitable domain D (field F) relatively to any given abstract group G of order $m+1$ $(m=1, 2, \cdots)$ represented as a regular group of configurational sets of dyads on $m+1$ elements. Two dyads $\alpha_i\alpha_j$ and $\alpha_k\alpha_l$ are then equivalent if and only if they occur in the same configurational set of dyads. Multiplication is determined by $\alpha_i\alpha_j\times\alpha_j\alpha_k=\alpha_i\alpha_k$ and by the preceding equivalences. Other instances of dyadic representation of linear algebras are given by two examples: 1. $X_2(\alpha)=x_1\alpha_1\alpha_1+x_2\alpha_1\alpha_2$ with equivalences $\alpha_1\alpha_1=\alpha_2\alpha_2$, $\alpha_1\alpha_2=-\alpha_2\alpha_1$. 2. $X_4(\alpha)=\sum x_j\alpha_1\alpha_j$ $(j=1, 2, 3, 4)$ with equivalences corresponding to those given by the author (Math. Ann. vol. 69 p. 584). In both examples multiplication is determined by $\alpha_i\alpha_j\times\alpha_j\alpha_k=\alpha_i\alpha_k$ and by the associated equivalences. (Received July 11, 1946.)

278. M. C. Sholander: *On the existence of the inverse operation in certain spaces.*

In a set S of elements $x, y, \cdots$ which admits a binary operation—here denoted by multiplication—an element a will be called regular if both (i) $ax=ay$ implies $x=y$ and (ii) $xa=ya$ implies $x=y$. An element a will be called proper if for each element b in S there exist unique solutions x and y in S for the equations $ax=b$ and $ya=b$. It is well known that if the multiplication is commutative and associative S can be imbedded in a space S' of the same type in such a way that all elements regular in S are proper in S'. In this paper it is shown the imbedding process can also be carried out in case the multiplication is one satisfying the alternation law $(ab)(cd)=(ac)(bd)$ and in case the regular elements of S are closed under multiplication. Thus if all elements of S are regular, S' is a quasi-group of a type studied, for example, by D. C. Murdoch (Trans. Amer. Math. Soc. vol. 49 (1941) pp. 392–409) and R. H. Bruck (Trans. Amer. Math. Soc. vol. 55 (1944) pp. 19–52). Various conditions which insure the necessary closure property in S are given in the paper. (Received July 26, 1946.)

279. J. M. Thomas: *Eliminants.*

If $R(a, b)$ denotes the resultant to two polynomials $x(t)$, $y(t)$ whose constant terms are a, b, the polynomial $R(a-x, b-y)$ in the two indeterminates x, y is the *eliminant* $E(x, y)$ of $x(t)$, $y(t)$. This paper (i) proves $E(x, y)=f^k$, where f is an irreducible polynomial and k is a positive integer; (ii) proves $E(x, y)$ is reducible $(1<k)$ if and only if $x(t)$, $y(t)$ are also polynomials in a second parameter which is itself a polynomial of degree at least two in t; (iii) expresses in terms of $E(x, y)$ algebraic conditions that a single polynomial $y(t)$ be a polynomial in $x(t)$ of degree k, where $1<k<\deg y$ (these last polynomials have been called by Ritt composite polynomials, Trans. Amer. Math. Soc. vol. 23 (1922) pp. 51–66). (Received July 27, 1946.)

280. J. H. M. Wedderburn: *Note on Goldbach's theorem.*

It is shown in this note that, if p and q are primes and $r=(p+q)/2$ is a prime, then $p-q$ is a multiple of 12 unless r has the form $2p-3$ or, when unity is reckoned as a prime, also $r=2p-1$. The proof is elementary and depends on reducing modulo 12. Similar statements apply if q is replaced by $-q$. (Received July 30, 1946.)

Analysis

281. R. P. Agnew: *Methods of summability which evaluate sequences of zeros and ones summable C_1.*

It is shown that if A is a method of summability belonging to a familiar general class, and if each divergent sequence of zeros and ones summable C_1 is summable A to the same value, then A must be regular. (Received July 13, 1946.)

282. R. P. Agnew: *Subseries of series which are not absolutely convergent.*

If $a(1)+a(2)+\cdots$ is a series of real or complex terms which fails to converge absolutely, then there is an increasing sequence $1 \leqq n_1 < n_2 < n_3 < \cdots$ of integers such that $n_{p+1}-n_p \rightarrow \infty$ as $p \rightarrow \infty$ and the subseries $a(n_1)+a(n_2)+\cdots$ is divergent. (Received July 13, 1946.)

283. H. P. Atkins: *On fractional derivatives of univalent functions.*

The following theorem is proved: If $f(z)$ is analytic and univalent in the unit circle with $f(0)=0$ and $f'(0)=1$, and if a is any real number and $n+1$ the smallest non-negative integer greater than a, then (without restriction if $a \leqq 3$, but assuming the Bieberbach conjecture if $a>3$) $\left|{}_0D_z^a f(z)-\sum_{j=0}^{m} {}_0D_z^{a-j} f^{(j)}(0)\right| \leqq (1-|z|)^{-a-2}\Gamma(a+1) \cdot \{aI(n+1-a, a+2)+|z| I(n-a, a+2)\}$ for $z \neq 0$. Equality is attained for real positive z by the function $f(z)=z(1-z)^{-2}$. The sum from 1 to n is to be interpreted as 0 if $n<1$, and ${}_0D_z^{a-j} f^{(j)}(0)$ means the derivative of order $a-j$ of the constant $f^{(j)}(0)$. The function $I(p, q)$ as used above represents the ratio of the incomplete beta function from 0 to $|z|$ to the corresponding complete beta function. A correspondingly weaker relation may be obtained using the Littlewood inequality instead of the Bieberbach conjecture. The theorem is a generalization of a relation for $a=0$ and $a=1$ due to G. Pick and proved for the remaining positive integers by F. Marty (*Sur les dérivées d'une fonction univalente*, C. R. Acad. Sci. Paris vol. 194 (1932) p. 1308). (Received June 19, 1945.)

284. J. D. Bankier: *Modified regular continued fractions.* Preliminary report.

If the algorithm for the expansion of a real number, y_0, into a regular continued fraction (r.c.f.), is modified by requiring that b_n $(n=0, 1, 2, \cdots)$ be either the integer nearest to y_n, or, (1) if $y_n=I+1/2$, where I is an integer, I or $I+1$ according as y_n is positive or negative, the resulting expansion is called a *modified* regular continued fraction (m.r.c.f.). The m.r.c.f. expansion terminates, if y_0 is rational, and does not have more elements than the corresponding r.c.f. If y_0 is irrational, its m.r.c.f. expansion converges to the value y_0 at least as rapidly as the corresponding r.c.f. Necessary and sufficient conditions that a continued fraction be a m.r.c.f. are that its elements be integers such that (2) $|b_n| \geqq 2$, (3) if $|b_{n-1}|=2$, $b_{n-1}b_n>0$ $(n=1, 2, 3, \cdots)$ and (1) holds if the continued fraction terminates. A m.r.c.f. is periodic if and only if it converges to a quadratic irrational. A method is given for obtaining the m.r.c.f. expansion of a number directly from the corresponding r.c.f. (Received July 2, 1946.)

285. Stefan Bergman: *The "extended class" in the theory of functions of two complex variables.*

Let $\mathfrak{M}^4$ be a domain with a distinguished boundary surface $\mathfrak{F}^2$. Let $H(z_1, z_2)$ be a (real) continuous function defined on $\mathfrak{F}^2$. As is well known, there does not exist in general a biharmonic function (real part of an analytic function of two complex variables) which is regular in $\mathfrak{M}^4$ and equals $H(z_1, z_2)$ on $\mathfrak{F}^2$. Let $\mathcal{V}=\mathcal{V}(\mathfrak{M}^4, H)$ be the totality of functions which are biharmonic in open $\mathfrak{M}^4$, continuous in closed $\mathfrak{M}^4$, and such that for every $B \in \mathcal{U}$, $B = H$ on $\mathfrak{F}^2$. Let $U(z_1, z_2) = \lim\inf B(z_1, z_2)$, $B \in \mathcal{U}$, $(z_1, z_2) \in \mathfrak{M}^4$. Similarly, let $\mathcal{L}$ denote the totality of biharmonic functions for which $B = H$ on $\mathfrak{F}^2$ and $L(z_1, z_2) = \lim\sup B(z_1, z_2)$, $B \in \mathcal{L}$. Under certain assumptions on H, it is possible to show that $U = L$ on the boundary $\mathfrak{m}^3$ of $\mathfrak{M}^4$. A harmonic function of four real variables which assumes on $\mathfrak{m}^3$ these values is denoted as a function of M-extended class belonging to H. Various applications to the theory of value distribution of functions of two complex variables and to the theory of pseudo-conformal mapping are made. (Received July 15, 1946.)

286. J. P. Brown: *Cyclic characteristic curves of a system of differential equations.*

Geometrical properties of the *cyclic characteristic curves* (*c.c.c.*), and in particular of the limit cycles of Poincaré, corresponding to the periodic solutions of $\{dx/dt = X(x, y);\ dy/dt = Y(x, y)\}$ are studied. Hurewicz's results (*Ordinary differential equations in the real domain*, Brown University, 1943) on the topology and stability of c.c.c. and of other limit sets of characteristic curves are restated. A *curve without contact* (*c.w.c.*) is defined to be a simple closed curve with continuously turning tangent, passing through no singularity of the vector field (X, Y) and nowhere tangent to (X, Y). There are two types of c.w.c., according as the vector (X, Y) points inside or outside the c.w.c. A c.c.c. is defined to be *persistent* if small arbitrary changes of X and Y change its position arbitrarily little. A c.c.c. is found to be persistent if and only if there exist c.w.c. of both kinds arbitrarily close to it. This result is interpreted in terms of the existence of limit cycles arbitrarily close to the c.c.c. Analogous definitions are set up and results obtained for the *stable persistence* of a c.c.c. (Received July 19, 1946.)

287. C. C. Camp: *Integral equations with kernels having discontinuities along two diagonals.*

If the kernel of $u(x) = \lambda\int_0^1 K(x, y)u(y)dy$ is the smallest of the quantities x, y, $1-x$, $1-y$, its partial derivatives have discontinuities along $y = x$ and $y = 1-x$. For this simple case $u(x) = C \sin (2k+1)\pi x$. An equivalent integro-differential system has been used to extend the theory to kernels which are more general. For example, if the kernel is normalized, has symmetry with respect to the line $x = 1/2$, satisfies the condition of a constant jump in K_x across the secondary diagonal, and if $K_{xx}(x, y)$ multiplied by a certain constant possesses a reciprocal kernel, then it may be shown that the usual expansion theorem holds for a function $f(x)$ such that $f(x) = f(1-x)$ (Received July 15, 1946.)

288. J. A. Clarkson: *A characterization of C-spaces.*

The set $C(T)$ of all bounded continuous real-valued functions $x(t)$ defined over a topological space T is a real Banach space. The following conditions are shown to be necessary and sufficient for a given real Banach space B to be equivalent to a space of this kind (that is, a "C-space"). (Space B is not assumed to be a ring, or partially ordered.) (1) S, the surface of the unit sphere in B, contains an element v such that if x is any element of S, one of the two segments $[x, v]$, $[x, -v]$ lies in S. (2) Let $\mathcal{C}$ be the set of all points x such that x lies on a ray terminating at v and passing through another point of S. Then $\mathcal{C}$ has the property that the intersection of two of its translates is again a translate. The proof utilizes results of Kakutani concerning partially ordered Banach spaces (*Concrete representation of abstract (M)-spaces*, Ann. of Math. vol. 42 (1941) pp. 994–1024). (Received July 6, 1946.)

289. H. V. Craig: *On the structure of intrinsic derivatives.*

The primary purpose of the present paper is to express the Mth order iterated intrinsic derivative of a higher order absolute tensor as a contraction. A method is developed for constructing extensors from higher order tensors by differentiating with respect to the curve parameter. Contraction of these derived extensors with certain generalized components of connection yields the Mth order iterated intrinsic derivative of the original tensor. The iterated intrinsic derivatives considered are the tensors obtained by repeatedly applying the classical intrinsic differentiation process of tensor analysis. (Received July 15, 1946.)

290. Bernard Epstein: *Integral representations of solutions of certain classes of partial differential equations.*

The method of integral operators developed by Bergman (see especially Trans. Amer. Math. Soc. vol. 53 and recent volumes of Duke Math. J.) is employed to study singularities of solutions of certain classes of elliptic equations in three variables, in particular the existence of solutions having singularities only along a sequence of prescribed closed curves. As a preliminary step, a theorem for harmonic functions of three variables similar to the Mittag-Leffler theorem of function theory is established. Then the use of the integral operator method leads directly to an analogous theorem for solutions of the above-mentioned more general types of equations. Finally, a similar theorem is obtained for solutions of a certain class of nonlinear equations. (Received July 8, 1946.)

291. N. J. Fine: *On the Walsh functions.* Preliminary report.

In 1923 Walsh completed the Rademacher system and studied some of its properties, pointing out the great similarity between the completed system and the trigonometric system. It is shown in the present paper that this similarity is not accidental, but is based on the fact that each system may be regarded as the character group of a certain compact commutative group. Some of the differences are shown to stem directly from the topological and algebraic structures of the underlying groups. The behavior of the Fourier expansion of a given function in terms of the Walsh functions is studied in more detail, and certain conditions sufficient for convergence are obtained. Questions of $(C, 1)$ and Abel summability are taken up for certain classes of functions. A discussion of the Lebesgue constants is given, covering their recursive and explicit representations, their generating function, and their maximal and average orders. The Riemann theory for general Walsh series is developed, with some important differences in method. It is proved that a Walsh series which converges everywhere to an integrable function is the Fourier series of its sum. (Received July 24, 1946.)

292. A. W. Goodman: *On some determinants related to p-valent functions.* Preliminary report.

The elementary inequality $|\sin n\theta/\sin \theta| \leqq n$ is generalized to give exact upper bounds for the ratios of certain pth order determinants involving $\sin n\theta_i$,

$i=1, 2, \cdots, p$. Exact upper bounds are also obtained for analogous expressions involving $\cos n\theta_i$ which are generalizations of $|\cos n\theta| \leqq 1$. A similar result is obtained for the ratios of certain modified Vandermonde determinants. Bieberbach's conjecture that $|b_n| \leqq n|b_1|$ for univalent functions is generalized to the set of functions p-valent in the unit circle. It is shown that the first set of trigonometric inequalities has a relation to this generalized conjecture. (Received July 11, 1946.)

293. S. H. Gould: *Lagrange multipliers and functional determinants.*

A geometric interpretation is given of the Lagrange multiplier rule, which is shown to be identical with the Jacobian rule for functional dependence with substitution of the words "stationary at a point" for "constant in a neighborhood." (Received July 12, 1946.)

294. R. N. Haskell: *Mean values and biharmonic polynomials.*

A class of functions is studied which at the centers of regular polygons are equal to certain linear combinations of the areal and peripheral means of the functions on the polygons. Let $P_n(x_0, y_0; r; \psi)$, $|P_n(x_0, y_0; r; \psi)|$ and $|p_n(x_0, y_0; r; \psi)|$, or for brevity P, $|P|$ and $|p|$, be the closed finite region, its area and the length of the boundary respectively of the regular n-gon $p_n(x_0, y_0; r; \psi)$ whose center is at (x_0, y_0). The letters r and ψ denote respectively the radius of the inscribed circle and the smallest positive angle made by any diagonal of the polygon with the positive x-axis. A typical result of the paper is the following. Given a function $f(x, y)$ summable superficially in the interior of a finite simply connected domain D, n a fixed integer, $n \geqq 3$, and ψ fixed, then a necessary and sufficient condition that $f(x_0, y_0) = 2|P|^{-1}\int\int_P f d\xi d\eta - |p|^{-1}\int_p f ds$ hold at each point (x_0, y_0) of D, and for all $p_n(x_0, y_0; r; \psi)$ in D for which $\int_p f ds$ exists, is that $f(x, y)$ shall be a biharmonic polynomial of degree at most $n+2$. (Received July 15, 1946.)

295. M. H. Heins: *On the Phragmén-Lindelöf principle.*

The problem has been proposed by Ahlfors (Trans. Amer. Math. Soc. vol. 41) to determine under the hypotheses of the classical Phragmén-Lindelöf principle for a half-plane whether or not $\lim_{r\to+\infty}(\log M(r))/r$ exists in all cases, the notation being that conventionally used (cf. Nevanlinna, *Eindeutige analytische Funktionen*, p. 43). This question is answered affirmatively. The limit exists and is never finite and strictly negative. The proof is based upon the following lemma: Let $f(z)$ denote a function which is analytic and of modulus not exceeding one for $R(z)>0$, let ϵ denote a given positive number, and let $E(r, \epsilon)$ denote for given r the set of θ for which $\log |f(re^{i\theta})| < -\epsilon r$. If $\limsup_{r\to+\infty}$meas. $E(r, \epsilon)>0$, then there exists a strictly positive constant κ such that $|f(z)| \leqq \exp[-\kappa R(z)]$ for $R(z)>0$. This lemma admits generalization in many directions and taken with its generalizations serves as a useful instrument for investigating the properties of the class of functions which are admitted by the hypotheses of the Phragmén-Lindelöf principle and in addition satisfy $\lim_{r\to+\infty}(\log M(r))/r < +\infty$. (Received June 10, 1946.)

296. J. G. Herriot: *On the restricted summability of multiple Fourier series.* Preliminary report.

Consider the Fourier series of an integrable function of k variables, periodic of period 2π in each variable. In a previous paper (Duke Math. J. vol. 11 (1944) pp.

735–754) the author showed that if $k \geqq 3$, restricted Abel summability and many restricted multiple Nörlund methods including restricted $(C, \alpha, \beta, \cdots, \kappa)$, $\alpha, \beta, \cdots, \kappa \geqq 0$, applied to such Fourier series do not possess the localization property. Bochner (Trans. Amer. Math. Soc. vol. 40 (1936) pp. 175–207) showed that the same result is true for restricted Riesz summability of type $\lambda_n = n$ and any positive order. It is now shown that restricted Riesz summability of type $\lambda_n = n^2$ and non-negative integral order δ applied to such Fourier series does not possess the localization property for $\delta \leqq k-2$. The condition $\delta \leqq k-2$ is necessary at least for $k \leqq 5$ since it is shown that restricted Riesz summability of type $\lambda_n = n^2$ and non-negative integral order δ applied to these Fourier series does possess the localization property for $\delta \geqq k-1$ provided $\delta \leqq 4$. This contradicts a remark of Bochner that these methods possess the localization property if $\delta > k$ but not if $\delta < k$. It is conjectured that the restriction $\delta + 4$ is not essential. (Received July 10, 1946.)

297. H. K. Hughes and Cleota G. Fry: *Asymptotic developments of types of generalized Bessel functions.*

Methods already developed by W. B. Ford, Newsom, and Hughes are employed to obtain general theorems giving the asymptotic behavior in the neighborhood of $z = \infty$ of the entire functions $f_1(z) = \sum_{n=0}^{\infty} z^n / \Gamma(kn+q)\Gamma(ln+p)$ $(l>0; l+k>0)$ and $f_2(z) = \sum_{n=0}^{\infty} z^n \Gamma(kn+q)/\Gamma(ln+p)$ $(l>k>0)$, where p and q are any suitable real or complex constants. From these theorems follow the asymptotic expansions of several important functions of Bessel types. One of these functions is the "generalized Bessel function" $j(z) = \sum_{n=0}^{\infty} z^n/n!\,\Gamma(kn+q)$, $k>-1$. (Received July 13, 1946.)

298. D. C. Lewis: *On polynomial interpolation and extrapolation by least squares with arbitrary weight function.*

Let $G_n^{(m)}(x, t)$ be defined as that polynomial in x of degree not greater than n which best fits (in the sense of least squares with distribution, $\alpha(x)$, having at least $n+1$ points of increase, on the interval $a \leqq x \leqq b$) the function $g^{(m)}(x, t)$ defined as follows: $g^{(m)}(x, t) \equiv (x-t)^m$ for $a \leqq x \leqq t$, and $g^{(m)}(x, t) \equiv 0$ for $t < x \leqq b$. Let $f(x)$ be any continuous function (defined for $A \leqq x \leqq B$, with $A \leqq a < b \leqq B$) with continuous derivatives up to and including the one of order $m-1$, and let the mth derivative $f^{(m)}(x)$ exist almost everywhere (including the points $x=a$ and $x=b$, where it is also continuous) and let $f^{(m)}(x)$ be of bounded variation. Let $P_n(x)$ be the polynomial of degree not greater than n which best fits $f(x)$ on the interval $a \leqq x \leqq b$ with respect to the same distribution $\alpha(x)$. It is proved in this paper that the following formula holds at every point on the interval $A \leqq x \leqq B$ with $1 \leqq m \leqq n$: $f(x) = P_n(x) + (1/m!)\int_b^x (x-t)^m df^{(m)}(t) + (1/m!)\int_a^b G_n^{(m)}(x, t) df^{(m)}(t)$. Certain generalizations of this theorem are also proved. (Received July 8, 1946.)

299. Szolem Mandelbrojt and G. R. MacLane: *On functions holomorphic in a strip region and an extension of Watson's problem.*

Let Δ be a region in the $s = \sigma + it$ plane given by $\sigma > a$ $(a \geqq -\infty)$, $|t| < g(\sigma)$, where $g(\sigma)$ is of bounded variation in (a, ∞) and $\lim_{\sigma \to \infty} g(\sigma) = \pi/2$. Let $N(\sigma)$ be an increasing function, and let $S(\sigma) = \pi/2 \int_c^\sigma (1/g(x))dx$, $c > a$. A necessary and sufficient condition that there exist a function $F(s)$, not identically zero, holomorphic in Δ, continuous in the closure of Δ, and satisfying the inequality $\log|F(\sigma + ig(\sigma))| \leqq \log M(\sigma) \leqq -N(\sigma)$, where $M(\sigma) = \mathrm{Max}_{|t| \leqq g(\sigma)} |F(\sigma+it)|$, is that (1) $\int_c^\infty N(\sigma) e^{-S(\sigma)} d\sigma < \infty$. If moreover $g'(\sigma)$

exists with $|g'(\sigma)| < M$, $g'(\sigma_2)-g'(\sigma_1) > A(\sigma_2-\sigma_1)$ $(A<0,\ \sigma_2>\sigma_1)$ then if $F(s)\not\equiv 0$, $\int_c^\infty \log |F(\sigma+ig(\sigma))| e^{-S(\sigma)}d\sigma > -\infty$. If $M_n>0$ $(n\geqq 1)$ and if $\lim M_n^{1/n}=\infty$, then a necessary and sufficient condition (since $F(s)$ is holomorphic in Δ, continuous in the closure of Δ, and satisfies there the inequalities $|F(s)| < M_n e^{-n\sigma}$ $(n\geqq 1)$ implying that $F(s)\equiv 0$) is that (2) $\sum \exp\left[-S(\log(M^{\circ}_{n+1}/M^{\circ}_n))\right]=\infty$, where $\{M^{\circ}_n\}$ is the convex regularized sequence of $\{M_n\}$, regularized by logarithms. These theorems generalize classical theorems known for the case in which Δ is the strip $|t|<\pi/2$, $S(\sigma)=\sigma-c$, and (2) becomes $\sum M^{\circ}_n/M^{\circ}_{n+1}=\infty$. (Received June 11, 1946.)

300. P. R. Masani: *Laurent factorization of analytic functions in normed rings.*

Let D_1 and D_2 be regions in the complex plane, $D=D_1\cap D_2\neq 0$, and f be an analytic function on D which possesses an inverse, with values in a normed ring X. This paper shows that under suitable assumptions as to D_1 and D_2, f can be factored on D: $f(z)=f_1(z)\cdot f_2(z)$, where f_1 and f_2 are analytic and possess inverses on D_1 and D_2 respectively. This question, which is deep when X is not commutative, has been studied for matrix-valued functions by Plemelj, G. D. Birkhoff, and H. Cartan. In this paper their results are generalized and proved by product integration. It is first shown that every function f analytic inside a closed contour C and continuous on C admits the Cauchy product integral representation $\exp(2\pi i f(z)) = \mathcal{P}\!\oint [1+D_w\{\exp\int(\zeta-z)^{-1}f(\zeta)d\zeta\}\cdot dw]$. Here z is a point inside C, 1 is the unit of X, D_w denotes product differentiation, $\int$ stands for the additive integral along C from a fixed point to w, and $\mathcal{P}\!\oint$ stands for the product contour integral along C. From here on this proof is practically parallel to the classical proof of Laurent's theorem. (Received July 9, 1946.)

301. P. R. Masani: *Multiplicative Riemann integration in normed rings.*

The author studies Riemann product-integration of functions from a real interval with values in a normed ring X. Volterra, Schlesinger, and Rasch have treated continuous matrix-valued functions. Garrett Birkhoff has studied continuous functions in an infinite-dimensional nonlinear space. The present theory is completely general in that it includes discontinuous functions. This extension is not trivial for even functions discontinuous everywhere may be R-integrable; and the product of such functions need not be R-integrable. The following results are obtained: properly product integrable functions are bounded; additive and product integrability are equivalent; the class of R-integrable functions is not in general a ring, when X is infinite-dimensional; the Peano-series development is valid, if the integrand is R-integrable; the indefinite product integral and product primitive agree up to a constant factor, whenever both exist; the rule for product integration by parts holds if the functions involved are R-integrable; the familiar method of integration after change of variable is generally valid. (Received July 9, 1946.)

302. Josephine M. Mitchell: *Value distribution of a function of two complex variables in a domain with distinguished boundary surface.*

The author considers meromorphic functions $f(z_1, z_2)$ of two complex variables $z_k=x_k+iy_k$, $k=1, 2$, in a certain class of domains with distinguished boundary surface F^2, namely in $M^4=\sum_{|z_2|\leq 1}B^2(z_2)$ where $B^2(z_2^{(0)})$ is a domain in the plane $z_2=z_2^{(0)}$ bounded by a simple closed curve $b^1(z_2^{(0)})=E[z_1=h(z_2^{(0)}, \lambda),\ 0\leqq\lambda\leqq 2\pi]$, $h(z_2^{(0)}, 0)$

$=h(z_2^{(0)}, 2\pi)$. Domains with distinguished boundary surfaces were introduced by S. Bergman (Math. Zeit. vol. 39 (1934) pp. 76–94, 605–608). As an example cases where $h(z_2, \lambda)=\sum_{k=-n}^{n} A_k(z_2)e^{ik\lambda}$, $A_k(z_2)$ analytic in $|z_2|\leqq 1$, are discussed. The notion of density function $D(a, z_1, z_2)$ in M^4 with respect to $i^3=\sum_{|t_2|\leqq 1} b^1(z_2)$ of the surfaces $f(z_1, z_2)=a$ is introduced and bounds are derived for $D(a, z_1, z_2)$. Let K^2 be a manifold which lies in i^3: $K^2=\sum_{0\leqq\lambda\leqq a} C^1(\lambda)$, $0<a\leqq 2\pi$, where $C^1(\lambda)=E[z_1=h(z_2,\lambda), z_2=z_2(t,\lambda), 0\leqq t\leqq 1]$, $z_2(0)=0$, $|z_2(1)|=1$. By means of the Milloux theorem and a method indicated by Bergman (Travaux de l'Institut Mathématique de Tbilissi vol. 1 (1937) pp. 187–204) the author obtains theorems concerning the value distribution of f in K^2 in terms of $D(0, z_1, z_2)$, $D(\infty, z_1, z_2)$ and the maximum of $|f|$ on F^2. (Received July 10, 1946.)

303. George Piranian: *Infinite iteration of totally regular transformations*. Preliminary report.

The *core* $C(s_n)$ of a sequence s_n in the complex plane is the intersection of all convex sets that contain all finite and infinite limit points of the sequence. A transformation A is *totally regular* provided $C(s_n)\supset C(As_n)$ for every sequence s_n. For a totally regular transformation A the *central A core* $C(A^\infty s_n)$ of s_n is the intersection of the cores $C(A^k s_n)$ $(k=1, 2, \cdots)$. THEOREM. *If A and B are totally regular and $AB=BA$, the central cores $C(A^\infty s_n)$ and $C(B^\infty s_n)$ have a common point, for each sequence s_n.* The *essential central A core* of s_n is the intersection of all central X cores of s_n, where X ranges over the set of totally regular transformations for which $AX=XA$. (Received July 17, 1946.)

304. Harry Pollard: *The inversion of the transforms with reiterated Stieltjes kernels.*

It is shown that the transforms $\int_{0+}^{\infty} H_n(x, y)d\alpha(y)$, where $H_n(x, y)$ is the nth iterate of the Stieltjes kernel $(x+y)^{-1}$, can be inverted by a linear differential operator of infinite order. (Received July 12, 1946.)

305. Harry Pollard: *The representation of e^{-x^λ} as a Laplace integral.*

The inverse Laplace transform of $\exp(-x^\lambda)$, $0<\lambda<1$, is computed explicitly, using Post's theory of the Laplace transform (Trans. Amer. Math. Soc. vol. 32 (1930) pp. 723–781). (Received July 12, 1946.)

306. W. T. Reid: *Solutions of linear differential equations.*

This paper is concerned with criteria for a function $y(x)$ to be a "weak solution" in the sense of Bochner (Ann. of Math. vol. 47 (1946) pp. 202–212) of a differential equation $\mathcal{L}[y]+g(x)=0$, where $\mathcal{L}[y]$ is a homogeneous linear differential operator involving ordinary or partial derivatives whose coefficients satisfy suitable differentiability conditions, and $g(x)$ is a given integrable function. One criterion presented is an extension of the condition on weak convergence used by Bochner for equations $\mathcal{L}[y]=0$ with constant coefficients, while another involving difference quotients is closely related to previous results of the author (Duke Math. J. vol. 12 (1945) pp. 685–694) for ordinary linear differential equations. (Received July 12, 1946.)

307. I. M. Sheffer: *On k-periodic systems of linear equations.*

Let (1) $\sum_{j=0}^{\infty} a_{nj}x_{n+j}=c_n$ $(n=0, 1, 2, \cdots)$ be a system of linear equations, and let (2) $A_n(t) = \sum_0^\infty a_{nj} t^j$. The system (1) is k-periodic, if (3) $A_{j+sk}(t) \equiv A_j(t)$ $(j=0, 1, \cdots, k-1; s=0, 1, 2, \cdots)$. The following results are obtained for suitable conditions on $\{a_{nj}\}$ and $\{c_n\}$: (i) A k-periodic system can be transformed into a 1-periodic system (but the process is not reversible). (ii) A k-periodic system can be factored into a product of k-periodic systems each of simple type. (iii) If (suitably restricted) perturbation terms are added to a k-periodic system, a factorization into factors of simple type is again possible. (iv) The number of independent solutions of a k-periodic system is determined. (Received July 8, 1946.)

308. Abraham Spitzbart: *The mininum of a certain integral.*

Let $f(z)$ be analytic within the unit circle C: $|z| = 1$, continuous within and on C, and with the value $f'(\alpha) = A$ prescribed at a point $z = \alpha$ within C. It is shown that for any such function the minimum of the integral $\int_C |f(z)|^p |dz|$, $p \geqq 1$, is given by $2\pi |A|^p (1-|\alpha|^2)^{p+1} \{2(1+|\alpha|^2)\}^{1-p} [(p-1)|\alpha| + (|\alpha|^2 - p^2 + 2p)^{1/2}]^{p-2} [p + |\alpha|^2 - |\alpha| (|\alpha|^2 - p^2 + 2p)^{1/2}]$ if $p \leqq 1 + |\alpha|$, and by $2\pi |A|^p 2^{1-p} p^p (1-|\alpha|^2)(1-|\alpha|)^p$ if $p \geqq 1 + |\alpha|$. If $p = 1$, the first form applies and reduces to a result of Macintyre and Rogosinski (Mathematical Notes, Edinburgh Math. Soc., vol. 35 (1945) pp. 1–3). (Received July 12, 1946.)

309. W. J. Trjitzinsky: *Singular integral equations of the first kind and those related to permutability and iteration.*

This is a systematic study of the topics indicated in the title when the kernels involved are L_2 separately in each variable, a fact which renders these problems singular. Essentially two methods are used, of which one involves suitable types of spectral theory. (Received July 12, 1946.)

310. Y. W. Tschen: *Branch points and flat points of minimal surfaces in R^3.*

The object of this paper is to study the behavior of minimal surfaces (in R^3) in the neighborhood of a branch point and to find an expression for the Anzahl of the branch points and of the flat points, respectively, for minimal surfaces with boundaries and given topological structure. Let the surface be denoted by the real parts of $\phi_i(\omega)$, $i = 1, 2, 3$, with branch points defined by $\phi_i'(\omega) = 0$, $i = 1, 2, 3$. To each branch point B belong two integers $k > 1$ and $l \geqq 1$: in the neighborhood of B the surface has k sheets and the normal vectors (with components ξ_i, $i = 1, 2, 3$) turn l times around the normal at B. Self-intersection of the surface at B is discussed. Branch points and flat points together are the points where the analytic function (of ω) $\xi_1\phi_1'' + \xi_2\phi_2'' + \xi_3\phi_3'' = 0$. From this is derived an analytic function which vanishes only at the branch points. Regularity of the functions on the boundaries is assumed in order to form the Anzahl by integration. Specific results are derived for polygon boundaries. (Received July 11, 1946.)

311. František Wolf: *On the continuation of analytic functions.*

T. Carleman has proved that if $x \in (a, b) \to \lim_{y \to 0} [f_1(x+iy) - f_2(x-iy)] = 0$ uniformly, then $f_1(z)$, $f_2(z)$ are parts of the same analytic function $f(z)$ which is analytic also on the open segment (a, b). If one drops the uniformity the same is true, except that $f(z)$ is analytic only in an everywhere dense set of intervals in (a, b). If the additional condition $f_1(x+iy)$, $f_2(x-iy)=o(1/y^n)$ is introduced, then $f(z)$ may have on (a, b) only poles and limit points of them of an even order less than n. The set of poles forms a set, of which no component is dense in itself. (Received July 29, 1946.)

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312. Hillel Poritsky and D. W. Dudley: *Conjugate action of involute helical gears with inclined axes.*

By an involute helical surface is meant a surface consisting of helices of a proper screw motion about an axis and such that a section by a plane perpendicular to the axis is an involute of a proper "base" circle. It is shown that two involute helical surfaces with inclined axes mate correctly in the sense that a uniform rotation of the one surface about its axis A is transformed into a uniform rotation of the other surface about its axis A', provided that certain inequalities in the distance and angle between A, A' are satisfied; the ratio of the two rotational velocities ω/ω' is independent of the distance between the two axes. The point of contact of the mating surfaces always moves along a fixed straight line in space with uniform velocity, this line being normal to each surface. Certain industrial applications are briefly discussed. (Received July 23, 1946.)

313. William Prager: *On the variational principles of plasticity.*

In earlier papers the author outlined a new mathematical theory of plasticity (Prikladnaia Matematika i Mekhanika N.S. vol. 5 (1941) pp. 419–430) and discussed certain variational principles associated with this theory (Duke Math. J. vol. 9 (1942) pp. 228–233). These variational principles were established under the assumption that the velocities or rates of stressing prescribed at the surface produce "loading" throughout the body. In the present paper this restriction is dropped and the general variational principles associated with the new theory of plasticity are established. (Received July 15, 1946.)

314. S. S. Shü: *On Taylor and Maccoll's equation of a cone moving in the air with supersonic speed.*

When an infinite cone is moving in the air with supersonic speed, the shock phenomena occur. Taylor and Maccoll (Proc. Roy. Soc. London, Ser. A. vol. 139 (1933) pp. 278–311) considered a conical flow behind an oblique shock wave and deduced a nonlinear ordinary differential equation which was integrated numerically. The purpose of the present note is first to transpose the equation to the differential-integral form $-\lambda'(\theta)=S_w \exp \int_\theta^{\theta_w}((2+\lambda'^2)/\lambda')d\theta/\rho \sin\theta$ (where S_w is a constant, e^λ is the ratio of the radial component of the velocity of the flow to the speed of the gas if allowed to be discharged into a vacuum and $\rho^{\gamma-1}=1-e^{2\lambda}(1+\lambda'^2)$) for which the author assumes the position and the intensity of the shock wave known and for which successive approximations are applied. In some cases, the sequence generated by the successive approximations is proved to be monotone and equi-continuous and therefore it converges uniformly to a solution of the problem in the large. A method is suggested for the practical calculation of the angle of the solid cone. The first approximation in which only the rational forms of elementary functions are involved gives a fairly good coincidence with Taylor and Maccoll's calculations. (Received July 20, 1946.)