315. J. L. Synge: *Approximations in elasticity based on the concept of function space.*

A state of an elastic body is defined by a set of six stress components, given as functions of position throughout the body. Such a state defines a point or vector S in function space, without any implication that the equations of equilibrium or compatibility or the boundary conditions are satisfied. A metric in the function space is defined by means of the strain-energy function. If S is the solution to a problem in which surface stress is given, S' an arbitrary state satisfying the equations of equilibrium and the boundary conditions (but not the equations of compatibility), and S'' another arbitrary state satisfying the equations of compatibility (but not the equations of equilibrium or the boundary conditions), then S is situated on the intersection of a hypersphere determined by S' and a hyperplane determined by S''. The center C of this hypercircle may be regarded as the "best" approximation. Its energy-error is given through the radius R of the hypercircle, which may be calculated from the formula $R=2^{-1}|S'-S''(S'\cdot S'')/S''^2|$. The method can be extended by using a sequence of states S_1'', S_2'', $\cdots$, and may also be used when the surface displacement is given instead of the surface stress. (Received July 22, 1946.)

316. C. A. Truesdell: *On Behrbohm and Pinl's linearization of the equation of two-dimensional steady flow of a compressible adiabatic fluid.*

In a recent note Behrbohm and Pinl (Zeitschrift für angewandte Mathematik und Mechanik vol. 21 (1941) pp. 193–203) have achieved a new linearization of the potential equation of two-dimensional steady adiabatic compressible flow in generalization of the Minkowski linearization of the equation for minimal surfaces. The author shows that Behrbohm and Pinl's result is equivalent by a simple change of variable to the ordinary linearization by Legendre's transformation, that Behrbohm and Pinl's subsidiary condition on the variables is superfluous, and that hence two of their variables may be interpreted physically as components of the velocity vector. He shows that Behrbohm and Pinl's equation suggests immediately the classical solutions of Tschaplygin. He discusses the possibility of other separations of the variables, and concludes that it is unlikely that any exist. (Received July 12, 1946.)

317. Alexander Weinstein: *On the method of sources and sinks.*

This paper contains an extension of the method of sources and sinks. New types of flows are obtained by taking sources distributed on circumferences, disks and cylinders. The procedure requires a modification of several formulae given by Beltrami who failed to recognize that Stokes' stream function of a circumference is a many-valued function. (Received July 6, 1946.)

Geometry

318. L. A. Dye: *A Cremona transformation in S_3 defined by a pencil of quartic surfaces.*

An involutorial Cremona transformation in S_3 is defined by means of a pencil of quartic surfaces $|F_4|$ with γ_{11} of genus 14 and γ_5 of genus 2, $[\gamma_{11}, \gamma_5]=18$ as a base. The γ_5 lies on a quadric H which intersects $|F_4|$ in γ_5 and a system of twisted cubics $|C_3|$. The residual intersections of the bisecants of a C_3 cut from the associated F_4 are pairs of conjugate points in the involution I. The γ_5 is invariant, not fundamental, and each C_3 is parasitic. After five factors H are removed from the equation of a general homoloid there remains an S_{19}: $\gamma_{11}^5+35l_4+4\gamma_2^2$. The 35 quadrisecants of γ_{11} and the 4 conics which meet it in 8 points are fundamental curves of the second species. (Received July 11, 1946.)

319. Michael Goldberg: *Tubular linkages.*

Special cases of movable polyhedral linkages are derived. They are composed of two or more prismatic linkages. In particular, if two quadrilateral prismatic linkages are joined to form a movable linkage, the quadrilateral joint may be a plane parallelogram, a skew deltoid or a skew isogram (a skew quadrilateral in which the opposite sides are equal). Some of these are derivable from the Bricard deformable octahedron. (See G. T. Bennett, Proc. London Math. Soc. (2) vol. 10 (1911–1912) pp. 309–343.) Combinations, including toroidal configurations, are exhibited. (Received July 2, 1946.)

320. V. G. Grove: *On congruences and conjugate nets.*

This paper is concerned with a metric study of congruences and conjugate nets on a surface in euclidian space of three dimensions. The formulas convenient to the study are written in terms of arbitrary parameters of the surface, and in terms of certain normalized components of vectors in the tangents to the curves of the net. An invariant is found which plays a role similar to that played by the geodesic curvature. Geometric constructions of all congruences conjugate to a surface and of all normal congruences are found. (Received June 27, 1946.)

321. C. C. Hsiung: *Projective theory of surfaces and conjugate nets in four-dimensional space.*

The purpose of this paper is to establish a theory of the projective differential geometry of surfaces sustaining a conjugate net in space S_4 of four dimensions. By purely geometric determinations a completely integrable system of linear homogeneous partial differential equations in canonical form is introduced, defining a conjugate net in S_4 except for a projective transformation. Some invariants of the parametric conjugate net N_x on an integral surface S of these equations are calculated. If one uses the local power series expansions for S, the curves of N_x and the Laplace transformed surfaces S_{-1}, S_1, the equations in local coordinates of some canonical configurations for S and hyperquadrics having contact of various orders with the surfaces and the curves are obtained. Further, some special classes of conjugate nets are also studied. One of their characterizations may be stated as follows: Conjugate nets with equal and nonzero Laplace-Darboux invariants in S_4 are characterized by the property that there exists a proper hyperquadric (and therefore ∞^3 such hyperquadrics) having second order contact with both S_{-1}, S_1 at the Laplace transformed points x_{-1}, x_1 respectively. (Received July 5, 1946.)

322. Edward Kasner and John DeCicco: *Geometry of harmonic transformations.*

The authors study the geometry of the infinite set (H) of harmonic transformations of the plane. Any such correspondence is defined by a pair of independent harmonic functions. This set (H) does not constitute a group. It is proved that the only real groups contained in (H) are the conformal group, the affine group, and the subgroups of these. The set (H) is characterized by means of the Kasner circle of the polygenic function in the following manner. A point transformation T is harmonic if and only if the associated center transformation is direct conformal. Another characterization is as follows. If, by a point transformation T, every parallel pencil of lines in the transformed plane corresponds to an isothermal family of curves in the original plane, then T is harmonic. Conformalities are the only real maps whereby every pencil of lines and every concentric set of circles correspond to isothermal families. Finally the transformation theory of the third order differential elements at a point, which is induced by (H), is studied. The harmonic set (H) induces a twelve-parameter set S_{12} of the total fifteen-parameter group G_{15} which is induced by the group of arbitrary point transformations. (Received July 19, 1946.)

323. Fred Supnick: *On the packing of spheres.* I.

Let a set of circles $c_1, \cdots, c_n$ with radii $r_1 \leqq r_2 \leqq \cdots \leqq r_n$, respectively, be packed (tangent, non-overlapping, connected) into a strip of width, A, $A/2 \geqq r_n$, bounded by the parallel lines l_1 and l_2, such that if c_1 is tangent to l_1, and c_2 and c_3 tangent to c_1 and l_2, then c_2 and c_3 have no interior point in common. Then the smallest rectangle of width A containing $c_1, \cdots, c_n$ is obtained by the packing $\{ \cdots c_{2i} \cdots c_4c_2c_1c_3c_5 \cdots c_{2k-1} \cdots \}$ and the largest by $\{c_1c_nc_3c_{n-2} \cdots - \cdots c_{n-3}c_4c_{n-1}c_2\}$ where any two adjacent c's are tangent to each other and to opposite sides of the strip. Analogous theorems are proved for the packing of spheres into containers, in particular, into cylinders of circular or regular polygonal cross section. (Received July 13, 1946.)

324. M. L. Vest: *An involutorial space transformation associated with a $Q_{1,n}$ congruence.*

The plane n-ic r, a line s meeting r at an $(n-1)$-point A, and a pencil of surfaces $|F_m|:s^{m-2}g_{4m-4}$ are given. Through a generic point $P(y)$ there passes a single F or $|F|$. The unique line t belonging to the congruence and passing through $P(y)$ meets F a second time in one residual point $Q(x)$, image of $P(y)$ under the transformation thus defined. The residual base curve g of $|F|$ is of order $4m-4$ and is considered noncomposite. It is shown that r, s and g are fundamental curves of the involution and that A is a fundamental point of the second kind. (Received June 28, 1946.)

325. G. W. Walker: *Games of the checkers family in line, plane, and space.*

Many variants from classical checkers can be made by varying the rules of play, particularly the typical move of the typical piece. If the move is in one dimension only, one has linear checkers. In two dimensions, the move may be across an edge of the cell, or through a corner, or both, or there may be a more complicated move like the knight's move in chess. Games can be based on various combinations. Interesting new features appear. In space, using a suitable frame, several excellent cube checker games can be defined, with many interesting new features. The field of play may be a network of white or black cells, or a looser network of cells holding together by their corners, or the entire frame. Places of local safety like the familiar double corner, and other strategical features, appear in new forms. There are many possible kinds of cube checker games, pure, combination, and hybrid games, multiple games, interfering games, cyclical games, and others. The best have already proved more interesting to play than the classical checker game. The checkers family can also be extended to spaces of higher dimensions. (Received July 12, 1946.)

STATISTICS AND PROBABILITY

326. H. W. Becker: *Stirling's numbers of the third kind.*

These are defined by ${}^{x}_{c\pm}S_{\pm r+1}=(x\pm c\pm r)\cdot{}^{x}_{c\pm}S_{\pm r}+{}_{c-1\pm}S_{\pm r}$, in which the weighting coefficient combines the characteristics of the recurrences for the Stirling's numbers of the first and second kinds. However, tables of the four varieties $\pm\pm$ are not only matrix products of certain tables of the first two kinds, but also unexpectedly simple vector multiples of the table of binomial coefficients. Summed over c, these are expressible by polynomials which are Chrystal-Jordan factorials of Stirling's polynomials of the second kind, and have the remarkable property of a triple recurrence. For example, ${}^{x}_{+}S_{-r+1}=(1+(x))^{r+1}=({}_{+}S+x)_{r+1}={}^{x}_{+}S_{-r}+x\cdot{}^{x-1}_{+}S_{-r}={}^{x-1}_{+}S_{-r+1}+(r+1)\cdot{}^{x-1}_{+}S_{-r}={}^{x+1}_{+}S_{-r}+(x-r)\cdot{}^{x}_{+}S_{-r}$. The combinatorial interpretations are rather intricate in terms of either permutations or rhyme schemes. (Received July 15, 1946.)

327. David Blackwell: *Conditional expectation and unbiased sequential estimation.*

It is shown that $E[f(x_\alpha)E_\alpha y]=E(fy)$ whenever $E(fy)$ is finite, and that $\sigma^2(E_\alpha y)\leqq\sigma^2(y)$, with equality holding only if $E_\alpha y=y$, where $E_\alpha y$ denotes the conditional expectation of y with respect to the family of chance variables x_α. These results imply that whenever there is a sufficient statistic u and an unbiased estimate t, not a function of u only, for a parameter p, the function $E_u t$, which is a function of u only, is an unbiased estimate for p with variance smaller than that of t. A sequential unbiased estimate for a parameter is obtained, such that when the sequential test terminates after i observations, the estimate is a function of a sufficient statistic for the parameter with respect to these observations. A special case of this estimate is that obtained by Girshick, Mosteller, and Savage (Ann. Math. Statist. vol. 17 (1946) pp. 13–23) for the parameter of a binomial distribution. (Received July 5, 1946.)

328. Paul Boschan: *The consolidated Doolittle technique.*

The quadratic matrix notation is interpreted as a segment in a sequence of matrices wherein each successor matrix is augmented by a bordering row and column. Extension theorems based on this idea date back into the last century. The step from the original concept to one of higher order is also fruitful in discussing inverse matrices, specifically the inverse of a symmetric matrix. The symmetry of the matrix of normal equations for a set of multiple regression-coefficients is restored by adding the transpose of the column on the right side of the equations, that is, the co-variances with the dependent variable and the variance of the dependent variable itself. The inverse of this matrix can be constructed as a partial sum over a series of matrices. Each individual element of this series is in itself meaningful. The solution for the set of multiple regression coefficients relating the kth variable to the preceding $(k-1)$ variables is a column matrix. The product of this matrix with its transpose expressed in terms of the residual variance forms the kth term in the matrix series. The summation of the first n products yields the inverse matrix. This characteristic of the inverse can be used to great advantage in the standardization of elementary computational steps. (Received July 17, 1946.)