ON FUNCTIONS HAVING SUBHARMONIC LOGARITHMS

MAXWELL O. READE

Saks has shown that if $f(x, y)$ is subharmonic in a domain G and if $\mu(e)$ is the completely additive, non-negative function of Borel sets associated with $f(x, y)$, then

$$\lim_{\rho\to 0} \frac{8}{\rho^2}\left[\frac{1}{\pi\rho^2}\int\int f(x+\xi, y+\eta)d\xi d\eta - f(x, y)\right] = D_s\mu(x, y),$$

and

$$\lim_{\rho\to 0} \frac{4}{\rho^2}\left[\frac{1}{2\pi\rho}\int f(x+\xi, y+\eta)ds - f(x, y)\right] \qquad = D_s\mu(x, y)$$

hold almost everywhere in G [7].[1] Here the first integral is extended over all (ξ, η) such that $\xi^2+\eta^2<\rho^2$, the second integral is extended over all (ξ, η) such that $\xi^2+\eta^2=\rho^2$, and $D_s\mu(x, y)$ is the symmetric derivative of $\mu(e)$ at (x, y).

The main result of this paper is an analogue of Saks' result for continuous functions having subharmonic logarithms. For such functions $f(x, y)$, it is shown that if $\sigma(e)$ is the completely additive, non-negative function of Borel sets associated with $\log f(x, y)$, then

$$\lim_{\rho\to 0} \frac{4}{\rho^2}\left\{\left(\frac{1}{2\pi\rho}\int f(x+\xi, y+\eta)ds\right)^2 - \frac{1}{\pi\rho^2}\int\int f^2(x+\xi, y+\eta)d\xi d\eta\right\}$$
$$= f^2(x, y)D_s\sigma(x, y)$$

holds almost everywhere in G.

Let G denote a domain (non-null connected open set) in the x, y-plane, $D(x, y; \rho)$ the open circular disc with center at (x, y) and radius ρ, and $C(x, y; \rho)$ the boundary of $D(x, y; \rho)$. If $f(x, y)$ is continuous in G, then $f(x, y)$ is said to be *subharmonic* in G if and only if

$$f(x, y) \leqq A(f; x, y; \rho) \equiv \frac{1}{\pi\rho^2}\int\int_{D(x,y;\rho)} f(\xi, \eta)d\xi d\eta \tag{1}$$

holds for each $D(x, y; \rho)$ in G [4]. It is well known that (1) can be replaced by either [4]

Presented to the Society, April 28, 1944, under the title *On functions of class PL*; received by the editors May 1, 1946.

[1] Numbers in brackets refer to the bibliography at the end of the paper.

$$f(x, y) \leqq L(f; x, y; \rho) \equiv \frac{1}{2\pi\rho}\int_{C(x,y;\rho)} f(\xi, \eta)ds,$$

or

$$A(f; x, y; \rho) \leqq L(f; x, y; \rho).$$

An important subclass of the class of functions subharmonic in G consists of those functions having subharmonic logarithms. These functions, studied by Beckenbach and Radó [1], are defined as follows. A function is said to be of class PL in G if and only if (i) $f(x, y) \geqq 0$, (ii) $f(x, y) \not\equiv 0$, (iii) $\log f(x, y)$ is subharmonic in G. It is fundamental in the theory of functions of class PL in G that $f(x, y)$ is of class PL in G if and only if

$$A(f^2; x, y; \rho) \leqq [L(f; x, y; \rho)]^2$$

holds for each $D(x, y; \rho)$ in G [1].

If $f(x, y)$ has continuous partial derivatives of the second order in G, then $f(x, y)$ is subharmonic in G if and only if $\Delta f(x, y) \geqq 0$ in G, and $f(x, y)$ is of class PL in G if and only if

$$f^2\Delta \log f \equiv f\Delta f - \left(\frac{\partial f}{\partial x}\right)^2 - \left(\frac{\partial f}{\partial y}\right)^2 \geqq 0 \tag{2}$$

in G [4]. Here Δ is the Laplace operator

$$\Delta \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\cdot$$

If $f(x, y)$ is subharmonic in G, then Riesz [6] has shown that there exists a unique, completely additive, non-negative function $\mu(e)$ of Borel sets e (for which the closure $\bar{e} \subset G$) with the following property. If D is a subdomain of G, such that $\overline{D} \subset G$, then $f(x, y)$ has the representation

$$f(x, y) \equiv f(P) = -\frac{1}{2\pi}\int\int_D \log\frac{1}{PQ}\, d\mu(e_Q) + H(P), \qquad P \in D, \tag{3}$$

where $PQ = ((x-\xi)^2 + (y-\eta)^2)^{1/2}$, $H(P)$ is harmonic in D, and where the integral is a Stieltjes-Radon integral [8].

Since the *density* of $\mu(e)$ at (x, y) is defined by [8]

$$D_s\mu(x, y) \equiv \lim_{\rho\to 0} \frac{\mu[D(x, y; \rho)]}{\pi\rho^2} \tag{4}$$

(which is known to exist almost everywhere [8]), then Saks' result may be stated as follows. If $f(x, y)$ is subharmonic in G, and if $\mu(e)$ is the set function used in (3), then

$$\begin{aligned}(5)\qquad &\lim_{\rho\to 0}\frac{8}{\rho^2}\left[A(f;\, x,\, y;\, \rho) - f(x,\, y)\right] \\ &\qquad = \lim_{\rho\to 0}\frac{4}{\rho^2}\left[L(f;\, x,\, y;\, \rho) - f(x,\, y)\right] = D_s\mu(x,\, y)\end{aligned}$$

holds almost everywhere in G. Saks' proof of (5) depends upon the representation (3) for $f(x, y)$.

If $f(x, y) \geqq 0$ is continuous and subharmonic in G, then $f^2(x, y)$ is continuous and subharmonic in G [4]. Hence by the "representation theorem" of Riesz, noted above, there exist unique, completely additive, non-negative set functions $\mu(e)$ and $\nu(e)$, for $\bar{e}\subset G$, associated with $f(x, y)$ and $f^2(x, y)$, respectively. Then the following lemmas hold.

LEMMA 1.

$$\begin{aligned}(6)\qquad &\lim_{\rho\to 0}\frac{1}{\rho^2}\left[L^2(f;\, x,\, y;\, \rho) - A(f^2;\, x,\, y;\, \rho)\right] \\ &\qquad = \frac{f(x,\, y)D_s\mu(x,\, y)}{2} - \frac{D_s\nu(x,\, y)}{8}\end{aligned}$$

holds almost everywhere in G.

PROOF. It is well known [4] that $L(f;\, x,\, y;\, \rho)\to f(x,\, y)$ and $A(f^2;\, x,\, y;\, \rho)\to f^2(x,\, y)$ on $\bar{e}\subset D$, as $\rho\to 0$. The relation (6) now follows from (5) and the identity

$$\begin{aligned}&\frac{L^2(f;\, x,\, y;\, \rho) - A(f^2;\, x,\, y;\, \rho)}{\rho^2} \\ &\qquad = \left[L(f;\, x,\, y;\, \rho) + f(x,\, y)\right]\left[\frac{L(f;\, x,\, y;\, \rho) - f(x,\, y)}{\rho^2}\right] \\ &\qquad\quad - \frac{A(f^2;\, x,\, y;\, \rho) - f^2(x,\, y)}{\rho^2}.\end{aligned}$$

LEMMA 2. *If e is a Borel set, $\bar{e}\subset G$, then*

$$(7)\qquad \nu(e) = 2\iint_e f(P)d\mu(e_P) + 2\iint_e\left[\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2\right]dxdy.$$

PROOF. Let D be a subdomain of G, such that $\bar{D}\subset G$. It follows from the proofs of the representation (3) for subharmonic functions, given by Evans [2] and Riesz [6], that $\mu(e)$ and $\nu(e)$ may be obtained as follows. If iterated averages of $f(x, y)$ are defined as

$$A_2(f;\, x,\, y;\, \rho) \equiv A(A;\, x,\, y;\, \rho), \quad A_3(f;\, x,\, y;\, \rho) \equiv A(A_2;\, x,\, y;\, \rho),$$

then there exists a sequence $\{\rho_n\} \searrow 0$, as $n \to \infty$, such that the set functions

$$\mu_n(e) \equiv \iint_e \Delta A_3(f;\, x,\, y;\, \rho_n)\,dxdy, \qquad \bar{e} \subset D, \tag{8}$$

$$\nu_n(e) \equiv \iint_e \Delta[A_3(f;\, x,\, y;\, \rho_n)]^2\,dxdy, \qquad \bar{e} \subset D, \tag{9}$$

converge to $\mu(e)$ and $\nu(e)$, respectively; that is,

$$\lim_{n\to\infty} \mu_n(e) = \mu(e),$$

if e is open and μ-regular (that is, $\mu(\bar{e}-e)=0$) and

$$\lim_{n\to\infty} \nu_n(e) = \nu(e),$$

if e is open and ν-regular (that is, $\nu(\bar{e}-e)=0$).

Now if R is an *oriented* rectangle in D, and if R is both μ- and ν-regular, and if the substitution

$$A_3(f;\, x,\, y;\, \rho_n) \equiv \mathfrak{A}_n(x,\, y)$$

is made, then it follows from (8) and (9) that

$$\begin{aligned} \nu(R) &= \lim_{n\to\infty} 2 \iint_R \mathfrak{A}_n(P)\,d\mu_n(e_P) \\ &\quad + \lim_{n\to\infty} 2 \iint_R \left[\left(\frac{\partial \mathfrak{A}_n}{\partial x}\right)^2 + \left(\frac{\partial \mathfrak{A}_n}{\partial y}\right)^2\right] dxdy \end{aligned} \tag{10}$$

holds. However, Frostman has shown [3] that if R is μ-regular, then

$$\lim_{n\to\infty} \iint_R \mathfrak{A}_n(P)\,d\mu_n(e_P) = \iint_R f(P)\,d\mu(e_P),$$

and Evans has shown [2] that

$$\lim_{n\to\infty} \iint_R \left[\left(\frac{\partial \mathfrak{A}_n}{\partial x}\right)^2 + \left(\frac{\partial \mathfrak{A}_n}{\partial y}\right)^2\right] dxdy = \iint_R \left[\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2\right] dxdy;$$

so that (10) may be written

$$\nu(R) = 2 \iint_R f(P)\,d\mu(e_P) + 2 \iint_R \left[\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2\right] dxdy. \tag{11}$$

Now each open oriented rectangle R in D is the point set limit of a monotone increasing sequence $\{R_n\}$ of open μ- and ν-regular rectangles, such that [5]

$$\lim_{n\to\infty} \mu(R_n) = \mu(R), \tag{12}$$

$$\lim_{n\to\infty} \nu(R_n) = \nu(R). \tag{13}$$

Hence it follows from (11), (12) and (13) that (11) must hold for *all* open oriented rectangles R in D. By a familiar argument used in the theory of set functions [5, 7], it follows that (7) holds for all Borel sets in D.

Since D was an arbitrary subdomain of G, the lemma now follows.

COROLLARY.

$$D_s\nu(x, y) = 2\left[f(x, y)D_s\mu(x, y) + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2\right] \tag{14}$$

almost everywhere in G.

PROOF. The relation (14) follows from (4), (7) and the classic theorem due to Lebesgue on the derivation of integrals [8].

THEOREM 1.

$$\begin{aligned}\lim_{\rho\to 0} \frac{4}{\rho^2}\,[L^2(f; x, y; \rho) - A(f^2; x, y; \rho)] \\ = f(x, y)D_s\mu(x, y) - \left(\frac{\partial f}{\partial x}\right)^2 - \left(\frac{\partial f}{\partial y}\right)^2\end{aligned} \tag{15}$$

almost everywhere in G.

PROOF. (15) follows from (6) and (14).

In the following, it is assumed that $f(x, y)$ is also of class PL in G. Hence $f(x, y) \equiv \exp u(x, y)$, where $u(x, y)$ is continuous and subharmonic in G, with associated set function called $\sigma(e)$.

LEMMA 3. *If e is a Borel set, $\bar{e} \subset G$, then*

$$\begin{aligned}\mu(e) = \iint_e \exp \mu(P)d\sigma(e_P) \\ + \iint_e \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] \exp u(x, y)dxdy.\end{aligned} \tag{16}$$

PROOF. It is inherent in the proof of (3) given by Evans [2] and Riesz [6], that $\mu(e)$ and $\sigma(e)$ may be found as follows. If the definition

$$A_3(u;\, x,\, y;\, \rho_n) \equiv a_n(x,\, y)$$

is made, then there exists a sequence $\{\rho_n\} \searrow 0$, as $n \to \infty$, such that

$$\mu_n^*(e) \equiv \int\int_e \Delta[\exp\, a_n(x,\, y)]\, dxdy \tag{17}$$

and

$$\sigma_n(e) \equiv \int\int_e \Delta a_n(x,\, y)\, dxdy \tag{18}$$

converge to $\mu(e)$ and $\sigma(e)$, respectively; that is

$$\lim_{n\to\infty} \mu_n^*(e) = \mu(e) \tag{19}$$

for each open μ-regular set e, and

$$\lim_{n\to\infty} \sigma_n(e) = \sigma(e) \tag{20}$$

for each open σ-regular set e.

Now an argument similar to that used in the proof of Lemma 2 shows that (16) follows from (17)–(20).

COROLLARY.

$$\begin{aligned} D_s\mu(x,\, y) = {} & \exp\, u(x,\, y)\cdot D_s\sigma(x,\, y) \\ & + \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] \exp\, u(x,\, y) \end{aligned} \tag{21}$$

almost everywhere in G.

PROOF. The corollary follows at once from (4), (16) and the theorem of Lebesgue on the derivation of integrals.

THEOREM 2.

$$\lim_{\rho\to 0} \frac{4}{\rho^2}\, [L^2(f;\, x,\, y;\, \rho) - A(f^2;\, x,\, y;\, \rho)] = f^2(x,\, y) D_s\sigma(x,\, y) \tag{22}$$

almost everywhere in G.

PROOF. The theorem, which is an analogue of Saks' result (5), follows at once from (15) and (21).

The relations (5), (6), (15) and (22) are examples of "generalized Laplacians" [**7**, **9**]. For example, if $f(x, y)$ is sufficiently smooth in G, then (22) yields

$$\lim_{\rho\to 0}\frac{4}{\rho^2}\left[L^2(f;\,x,\,y;\,\rho)-A(f^2;\,x,\,y;\,\rho)\right]=f^2(x,\,y)\Delta\log f(x,\,y),$$

which bears an important relation to the defining inequality (2) for smooth functions of class PL.

Bibliography

1. E. F. Beckenbach and T. Radó, *Subharmonic functions and surfaces of negative curvature*, Trans. Amer. Math. Soc. vol. 35 (1933) pp. 662–674.

2. G. C. Evans, *On potentials of positive mass*, Trans. Amer. Math. Soc. vol. 37 (1935) pp. 226–253.

3. O. Frostman, *Potentiel d'équilibre et capacité des ensembles . . .*, Thesis, Lund, 1935.

4. T. Radó, *Subharmonic functions*, Berlin, 1937.

5. P. Reichelderfer and L. Ringenberg, *The extension of rectangle functions*, Duke Math. J. vol. 8 (1941) pp. 231–242.

6. F. Riesz, *Sur les fonctions subharmoniques et leur rapport à la théorie du potentiel*, Acta Math. vol. 54 (1930) pp. 321–360.

7. S. Saks, *On the operators of Blaschke and Privaloff for subharmonic functions*, Rec. Math. (Mat. Sbornik) N. S. vol. 9 (1941) pp. 451–456.

8. ———, *Theory of the integral*, New York, 1937.

9. N. Wiener, *Laplacians and continuous linear functionals*, Acta Univ. Szeged. vol. 3 (1926) pp. 7–16.

Purdue University