OPEN TRANSFORMATIONS AND DIMENSION[1]

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This paper considers separable metric spaces A and B and open transformations. If, for each $x \in A$, $f(x) \in B$ and the image under f of every open set in A is a set open in B, then f is an *open* transformation. Continuity of f is not assumed. Such transformations have been studied by Rhoda Manning [1].[2]

THEOREM 1. *If $f(A) = B$ where f is open, then there exists a subset A_1 of A such that* (1) *$f(A_1) = B$,* (2) *for $y \in B$, the set $f^{-1}(y) \cdot A_1$ is countable, and* (3) *f, considered as a transformation of A_1 into B, is open.*

PROOF. Let $K_1, K_2, \cdots$ denote the elements of a countable base (open sets) for the space A. For every $y \in B$ and each i let P_{yi} be a point of $K_i \cdot f^{-1}(y)$, provided this set is nonvacuous. Let A_1 be the set of all points P_{yi} so obtained. Properties (1) and (2) are obvious. To prove (3), let V be an open set in A_1, and U an open set in A such that $U \cdot A_1 = V$. Now for every y the set $f^{-1}(y) \cdot A_1$ is dense in $f^{-1}(y)$. Hence if $f^{-1}(y)$ has a point in U then it has a point in $A_1 \cdot U$ so $f(V) = f(U)$ is an open set in B.

THEOREM 2. *There exist countable-fold open mappings*[3] *which increase dimension.*

PROOF. There exist open mappings which increase dimension [2]. Theorem 2 follows by applying Theorem 1 to any such example.[4]

THEOREM 3. *If* $\dim A = n$ *and* $-1 < m \leqq n$, *then there exists a* B *and a transformation* f *such that* (1) $f(A) = B$, (2) f *is open and* 1-1, *and* (3) $\dim B = m$. *In other words, dimension can be lowered at will by a* 1-1 *open transformation.*[5]

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[1] One statement in the abstract (Bull. Amer. Math. Soc. Abstract 59-5-210) is incorrect. Theorem 4 gives the correct statement.

[2] Numbers in brackets refer to the bibliography.

[3] A *mapping* is a continuous transformation.

[4] Alexandroff [5] has proved that if A is *compact* then no countable-fold open mapping can increase dimension.

[5] Compare the following special case of a theorem of Hurewicz [4, p. 91, Theorem VI 7]): "If f is a *closed mapping* of A into B and for each $y \in B$, $f^{-1}(y)$ is zero-dimensional, then $\dim B \geqq \dim A$." In a footnote (loc. cit.) the authors state that it is not known if Theorem VI 7 is true for *open* mappings. The answer is in the negative and their example VI 10 is a counter example, as the mapping f is actually open.

PROOF. It is sufficient to prove the theorem for $m=n-1$. For if $f(A)=B$ and $g(B)=C$, where f and g are open and 1-1, then the product gf is open and 1-1 and transforms A onto C.

The author has shown[6] that there exists a space B_1 of dimension $n-1$, and an at most 2-to-1 mapping ϕ of B_1 onto A. Furthermore the subset of B_1 consisting of all x such that $\phi^{-1}\phi(x)=x$ is $(n-1)$-dimensional. For $y\in A$, if $\phi^{-1}(y)$ is a single point then write $f(y)=\phi^{-1}(y)$. If $\phi^{-1}(y)$ is double-valued, select arbitrarily one point in $\phi^{-1}(y)$ and define $f(y)$ to be this point. Let $f(A)=B$. Then B is of dimension $n-1$ and f, as inverse of a 1-1 continuous function, is open.

Remark. If $f(A)=B$ is open and 1-1, then f^{-1} is continuous and 1-1. Thus Theorem 3 provides examples of arbitrary increases of dimension by 1-1 mappings.

THEOREM 4. *Suppose $f(A)=B$ is open, B is locally compact, and for each $y\in B$ the set $f^{-1}(y)$ is not dense-in-itself. Then* $\dim B\leqq \dim A$.[7]

PROOF. Let $K_1, K_2, \cdots$ be an open base for A. For each i let A_i be the set of all $x\in K_i$ such that if $x'\in K_i$ and $f(x')=f(x)$, then $x'=x$, and let $B_i=f(A_i)$. For $y\in B$, $f^{-1}(y)$ contains an isolated point (with respect to $f^{-1}(y)$), so $B=\sum_{i=1}^{\infty}B_i$. We prove next that B_i is closed in the open set $f(K_i)$. For suppose on the contrary that there exists a sequence $y_n\rightarrow y$, where y and each y_n are in $f(K_i)$, $y_n\in B_i$ but $y\notin B_i$. Since $y\in f(K_i)$ but not in B_i, there exist distinct points x and x' in K_i such that $f(x)=f(x')=y$. For each n there is a unique $x_n\in K_i$ such that $f(x_n)=y_n$. Since $y_n\rightarrow y$ it follows that $\lim\inf f^{-1}(y_n)\supset f^{-1}(y)\supset(x+x')$. But $f^{-1}(y_n)$ has only the one point x_n in K. This gives a contradiction.

Now let $M_{i1}, M_{i2}, \cdots$ be closed and compact subsets of B with $M_{i1}+M_{i2}+\cdots=f(K_i)$. Write $B_{ij}=B_i\cdot M_{ij}$. Then B_{ij} is a compact space and $B=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}B_{ij}$. Hence if $\dim B_{ij}\leqq\dim A$ for all i and j, then $\dim B\leqq\dim A$ (see [4, Theorem III 2, p. 30]). Write $A_{ij}=A_i\cdot f^{-1}(B_{ij})$. Over A_{ij}, f is open and 1-1 to B_{ij}. For $y\in B_{ij}$ define $g(y)=x\in A_{ij}$ such that $f(x)=y$. Now g, as the inverse of any open 1-1 transformation, is continuous; thus g is a mapping and $g(B_{ij})=A_{ij}$. Furthermore B_{ij} is compact, so that g is a *closed mapping*. Also g is 1-1. Then[8] $\dim B_{ij}\leqq\dim A$, so $\dim B\leqq\dim A$.

Remark. In Theorem 4 the assumption that B is locally compact

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[6] See [3], especially Theorem 9.1. If the M of this theorem is the space A, then M_1 is the desired space B_1 and ϕ_1 is the desired mapping ϕ.

[7] See footnote 4.

[8] See [4, Theorem VI 7, p. 91].

can be replaced by the weaker assumption that every point of B has arbitrarily small neighborhoods with compact boundary. The proof is not given. Some assumption of compactness seems necessary. Consider the following example.

Example. There exist, in the plane, spaces A and B and an open 1-1 transformation f with $f(A)=B$, with dim $A=0$ and dim $B=1$.

The space B is an example due to Sierpiński [6, pp. 81–83]. This space B has the following properties: (1) it is 1-dimensional, (2) it lies in the plane with $0 \leqq x \leqq 1$, $0 \leqq y \leqq 1$, (3) it contains at most one point (x, y) for a given x, and (4) the set of all points $(x, 0)$, such that for some y the point $(x, y) \in B$, is homeomorphic to the Cantor ternary set. Let A be this set defined by (4). Then A is the projection of B onto the x-axis, and for $(x, 0) \in A$ there is a single point (x, y) in B. Define f as follows: $f(x, 0)=f(x, y) \in B$. Then f has the required properties.

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