SOME REMARKS AND CORRECTIONS TO ONE OF MY PAPERS

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Professor Hartmann pointed out two inaccuracies in my paper *Some remarks about additive and multiplicative functions* (Bull. Amer. Math. Soc. vol. 52 (1946) pp. 527–537) (see Mathematical Reviews vol. 7 (1946) p. 577).

His first objection is that my proof of Theorem 12 (see p. 535) assumes that $f(p^\alpha) \geqq 0$. The only place the error occurs is in the fifth formula line of p. 536. But the error is quite easy to correct, only a $O(1)$ term is missing. The correct version of the formula is

$$\sum_{m=1}^{n} g_k(m) \leqq n \sum_d \frac{h_k(d)}{d} + O(1) < n \prod_p \left(1 + \frac{h_k(p)}{p}\right) + O(1).$$

Otherwise the proof is unchanged.

His second objection is against Theorem 13 (pp. 536–537) and is more serious.

Theorem 13 was stated as follows: Let $g(n) \geqq 0$ be multiplicative. Then the necessary and sufficient condition for the existence of the distribution function is that

$$(1) \qquad \sum_p \frac{(g(p) - 1)'}{p} < \infty, \qquad \sum_p \frac{((g(p) - 1)')^2}{p} < \infty$$

where $(g(p) - 1)' = g(p) - 1$ if $|g(p) - 1| \leqq 1$ and 1 otherwise.

I try to prove this by putting $\log g(n) = f(n)$ and state that $g(n)$ has a distribution function if and only if $f(n)$ has a distribution function.

In his review Hartmann points out that first of all this implies $g(n) > 0$ (instead of $g(n) \geqq 0$), and in a letter he points out that my statement is incorrect if $g(n)$ has a distribution function but $\lim_{x \to +0} G(x) > 0$ ($G(x)$ being the distribution function of $g(x)$). (I seem to remember that in my mind I was somehow unwilling to admit these $G(x)$ as distribution functions, but neglected to state this.)

In fact it is easy to see that this case can occur. Put $g(p^\alpha) = 1/2$ for all p and α. Then $G(x) = 1$ for all $x \geqq 0$, but clearly $f(n)$ has no distribution function, and the series (1) do not converge. Thus Theorem 13 is incorrect as it stands. The correct version may be stated as follows:

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THEOREM 13′. *Let* $g(n) \geqq 0$ *be multiplicative. Assume that the series* (1) *converge. Then* $g(n)$ *has a distribution function. The converse is also true unless* $G(x) = 1$ *for all* $x \geqq 0$.

First of all we remark that if

$$\sum_{g(p)=0} \frac{1}{p} = \infty$$

we have $G(x) = 1$ for all $x \geqq 0$ (since almost all integers are divisible by a p with $g(p) = 0$). Thus this case can be neglected, and we can assume that the primes with $g(p) = 0$ can be neglected, since they do not influence the convergence of the series (1) or the existence of the distribution function.[1]

The first part of Theorem 13 follows as on p. 537 of my paper.

Next we investigate the converse. If we assume that $\lim_{x \to +0} G(x) = 0$ the convergence of (1) follows as on p. 537, since in this case it really is true that $g(n)$ has a distribution function if and only if $f(n)$ has a distribution function.

Assume now

$$\lim_{x \to +0} G(x) = c > 0. \tag{2}$$

We shall show $c = 1$. Suppose that $c < 1$, we shall show that this leads to a contradiction.

Denote by $F(x)$ the density of integers with $f(n) < x$ (where $f(n) = \log g(n)$). Clearly $F(x)$ exists and satisfies ($G(x)$ is a distribution function)

$$\lim_{x \to -\infty} F(x) = c > 0, \qquad \lim_{x \to +\infty} F(x) = 1 \qquad (c < 1). \tag{3}$$

From now on we make constant use of my joint paper with Wintner[1] (referred to as E.W.). It follows from (3) that there exist real numbers a and b such that

$$-\infty < a < b < \infty \quad \text{and} \quad F(b) - F(a) > 0. \tag{4}$$

From (4) and E.W. §9, p. 717 it follows that $|f(p)| < A$ (except for a sequence of primes q with $\sum 1/q < \infty$, which can be neglected).

Next we deduce (E.W. §3, pp. 714–715) that

$$\sum_{p} \frac{(f(p))^2}{p} < \infty. \tag{5}$$

[1] Amer. J. Math. vol. 61 (1939) pp. 713–721.

Further it follows that (E.W. §4, p. 714)

$$\left|\sum_{p<x} \frac{f(p)}{p}\right| < B \qquad (B \text{ independent of } x). \tag{6}$$

In §6, p. 716 it is shown that from $|f(p)| < A$, (4) and (5) it follows that

$$\sum_{m=1}^{n} (f(m))^2 < Cn. \tag{7}$$

But clearly (7) contradicts (3) (since (3) implies that the density of integers with $f(m) > D$ is not less than c for every D), which completes the proof of Theorem 13′.

The following question can be raised: Let $f(n)$ be additive and assume that for some $a<b$ the density of the integers satisfying $a \leqq f(n) \leqq b$ exists and is different from 0. Does it then follow that $f(n)$ has a distribution function?

By the same methods as just used we can show that

$$|f(p)| < c, \qquad \sum_{p} \frac{(f(p)')^2}{p} < \infty, \qquad \sum_{p} \frac{f(p)'}{p} < \infty.$$

But at present I cannot decide whether the distribution function has to exist.

Professor Hartmann also pointed out the following misprints in my previous paper:

(1) The first sentence of Theorem 12 should read "Let $f(p^\alpha) \leqq C$."

(2) The inequality symbol in the two formula lines at the bottom of p. 535 should be "$\leqq$" instead of "$>$."

(3) On p. 537, in the line following the third formula line "$(\log g(p))^1 > \cdots$" should be "$(\log g(p))^2 > \cdots$."

(4) On p. 537, the fifth formula line should be "$\sum(1/p) \cdots$" instead of "$\sum(1/2) \cdots$."

(5) In the next to the last line of the paper, p. 537, "$\cdots f(n)$" should be "$\cdots g(n)$."

(6) The first formula on p. 529 should read "$\cdots \exp\exp(d\phi(n))$" instead of "$\cdots \exp\exp(\phi(n))$."

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