RAMANUJAN'S FUNCTION $\tau(n)$—A CONGRUENCE PROPERTY

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In a recent paper[1] S. Chowla and I proved that

$$\tau(n) \equiv n\sigma_3(n) \pmod 7 \tag{1}$$

where

$$\sum_{n=1}^{\infty} \tau(n)x^n = x\prod_{n=1}^{\infty}(1 - x^n)^{24} \qquad (|x| < 1)$$

and $\sigma_k(n)$ is the sum of the kth powers of the divisors of n.

The following is another short proof of (1).

Writing

$$P = 1 - 24\sum_1^{\infty} \sigma(n)x^n,$$

$$Q = 1 + 240\sum_1^{\infty} \sigma_3(n)x^n,$$

and

$$R = 1 - 504\sum_1^{\infty} \sigma_5(n)x^n,$$

Ramanujan proved that

$$1728\sum_1^{\infty} \tau(n)x^n = Q^3 - R^2.$$

From relation 10, Table III, p. 142, Ramanujan's *Collected Papers*, Cambridge, 1927, we derive

$$\begin{aligned} 3(1728)\sum_1^{\infty} \tau(n)x^n &= 3(Q^3 - R^2) \\ &= 41472\sum_1^{\infty} n^4\sigma_3(n)x^n - 7R^2 \\ &\quad - 7(P^4Q - 4P^3R + 6P^2Q^2 - 4PQR). \end{aligned}$$

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[1] Sent to the Quarterly Journal of Mathematics, Oxford.

Therefore comparing coefficients of x^n, we have

$$3(1728)\tau(n) \equiv 41472n^4\sigma_3(n) \pmod 7$$

or

$$\tau(n) \equiv n^4\sigma_3(n) \pmod 7$$
$$\equiv n\sigma_3(n) \pmod 7,$$

for if[2]

$$(n/7) = -1, \qquad \sigma_3(n) \equiv 0 \pmod 7$$

and if $(n/7) = +1$, or $n \equiv 0 \pmod 7$

$$n^4 \equiv n \pmod 7.$$

Note: The results for moduli 5 and 691 can be similarly obtained by using Relations 5, Table II, and 6, Table I, respectively, of Ramanujan's *Collected papers*, pages 141–142.

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[2] This is a particular case of the more general theorem: If p is a prime, and if $(k/p) = -1$, then $\sigma_{(p-1)/2}(pt+k) \equiv 0 \pmod p$.