A THEOREM ON MONOTONE INTERIOR TRANSFORMATIONS

EDWIN E. MOISE

B. Knaster[1] has raised the question whether there is a compact metric continuum M, irreducible between two of its points, and a monotone interior transformation T, throwing M into the unit interval, such that for each x of $T(M)$, $T^{-1}(x)$ is an arc. In the present note, we shall answer this question in the negative.

Suppose that such a continuum exists, and let $T(M)=I=[0, 1]$. Let K be a subcontinuum of M which contains points of $T^{-1}(x)$ and $T^{-1}(y)$, where $x, y \in I$ and $x \neq y$. Since M is an irreducible continuum, K contains $T^{-1}(z)$ for each z between x and y; and since T is interior, K contains $T^{-1}(x)$ and $T^{-1}(y)$. It follows that each subcontinuum of M either is an arc or contains an open subset of M, but not both. In the first case, K lies in the inverse image of a point of I, and in the second case, K is the inverse image of a subinterval of I. In either case, K is decomposable.

Now let C_1 be a simple chain of open subsets of M, with links $c_1, c_2, \cdots, c_k$, covering $T^{-1}(0)$, such that each link of C_1 contains a point of $T^{-1}(0)$ which does not lie in the closure of the sum of the other links of C_1. There is a subcontinuum K_1 of M, lying in $\sum c_i$ and containing $T^{-1}(0)$, such that for each link c of C_1, each component of $K_1 - c\cdot K_1$ is a boundary subset of M; each such component is therefore an arc. Let K be $T^{-1}(I')$, $I' \subset I$. For each x of I', $\bar{c}_k$ is the sum of two mutually exclusive closed point-sets H and H', containing $\bar{c}_k T^{-1}(0)$ and $\bar{c}_k T^{-1}(x)$ respectively. In fact, for each $j<k$, the closure of $c_2+c_3+\cdots+c_j$ has a separation into closed sets which induces such a separation of $\bar{c}_k$. But the closure of[2] C_1^* obviously has no such separation; whence it follows that there is a component L of $K_1 - K_1\cdot\bar{c}_k$ which has a limit-point in H and a limit-point in H'. L must contain a point not in the closure of $c_2+c_3+\cdots+c_{k-1}$; and being a boundary set, L is an arc. L is therefore a subset of the inverse image of a point y of I. Let A_1 be $T^{-1}(0)$, and let A_2 be $T^{-1}(y)$.

By repeated application of the above procedure, we obtain a sequence $A_1, A_2, \cdots$ of arcs lying in M, and a sequence $C_1, C_2, \cdots$ of simple chains of open subsets of M, such that (1) C_i covers A_i, (2)

Presented to the Society, April 17, 1948; received by the editors June 10, 1948.

[1] B. Knaster, *Un continu irréductible à décomposition continue en tranches*, Fund. Math. vol. 25 (1935) p. 577.

[2] If C is a collection of sets, then C^* denotes the sum of the elements of C.

the closure of C^*_{i+1} lies in C_i^*, (3) the maximum diameter of the links of C_i is less than $1/i$, and (4) C_{i+1} contains two mutually exclusive chains, each of which has a link in each link of C_i.

Let N be the common part of the closures of the sets C_i^*. Suppose that N is the sum of two mutually exclusive closed point-sets H and H'. Condition (3) now implies that for some i, C_i is the sum of two mutually exclusive open sets each of which is a sum of links of C_i; since C_i is a simple chain, this is impossible. Therefore N is a continuum.

It is not difficult to show that N is indecomposable. The proof indicated below is very similar to a proof given in another connection by Knaster.[3] We wish to show that if N' is a proper subcontinuum of N, then every point of N' is a limit-point of $N-N'$. For each i, let C_i' be the set of all links of C_i that contain a point of N'. There is a k such that for $i>k$, C_i-C_i' contains two intersecting links of C_i. It follows that for $i>k$, C_{i+2} contains two mutually exclusive chains, one of which covers N' and both of which have a link in each link of C_{i+1}. Therefore N is the closure of $N-N'$. Since M, by hypothesis, contains no indecomposable continuum, the proof of the following theorem is now complete:

THEOREM. *For no compact, metric, irreducible continuum M is there a monotone interior transformation throwing M into an arc A such that the inverse image of each point of A is an arc.*

UNIVERSITY OF MICHIGAN

[3] B. Knaster, *Un continu dont tout sous-continu est indécomposable*, Fund. Math. vol. 3 (1922) p. 279.