THE SUMMER MEETING IN BOULDER

The fifty-fifth Summer Meeting and the thirty-first Colloquium of the American Mathematical Society were held at the University of Colorado, Boulder, Colorado, Tuesday to Friday, August 30–September 2, 1949. The Mathematical Association of America met on Monday and Tuesday, and the Econometric Society and the Institute for Mathematical Statistics held meetings Monday through Thursday. Attendance at the meetings was approximately 800, including the following 394 members of the Society:

J. C. Abbott, V. W. Adkisson, O. W. Albert, E. B. Allen, R. D. Anderson, T. W. Anderson, R. V. Andree, B. H. Arnold, K. J. Arnold, M. N. Arnoldy, W. L. Ayres. Reinhold Baer, E. W. Barankin, Joshua Barlaz, I. A. Barnett, D. Y. Barrer, M. A, Basoco, P. T. Bateman, F. D. Bateman, J. W. Beach, J. E. Bearman, E. F. Beckenbach, M. M. Beenken, A. K. Bell, J. H. Bell, P. O. Bell, B. C. Bellamy, C. A. Bennett, Theodore Bennett, A. H. Berger, Herman Betz, S. F. Bibb, F. C. Biesele, E. E. Blanche, W. A. Blankinship, H. D. Block, M. L. Boas, R. P. Boas, H. F. Bohnenblust, A. W. Boldyreff, F. E. Bortle, T. A. Botts, J. W. Bradshaw, H. E. Bray, J. R. Britton, J. C. Brixey, J. R. Buchi, C. C. Buck, E. F. Buck, R. C. Buck, C. E. Buell, P. B. Burcham, Herbert Busemann, J. H. Bushey, Jewell H. Bushey, S. S. Cairns, W. D. Cairns, J. W. Calkin, E. A. Cameron, C. C. Camp, H. H. Campaigne, B. G. Carlson, F. M. Carpenter, W. B. Carver, C. R. Cassity, T. E. Caywood, Abraham Charnes, Harold Chatland, L. G. Chelius, Herman Chernoff, K. L. Chung, R. V. Churchill, H. M. Clark, H. E. Clarkson, Nathaniel Coburn, C. J. Coe, Nancy Cole, L. A. Colquitt, E. G. H. Comfort, R. M. Cook, J. A. Cooley, Max Coral, N. A. Court, M. J. Cox, F. G. Creese, E. L. Crow, H. J. Curtis, J. H. Curtiss, P. H. Daus, A. C. Davis, E. A. Davis, A. S. Day, M. M. Day, R. F. Deniston, W. W. Denton, A. H. Diamond, R. P. Dilworth, Bernard Dimsdale, W. J. Dixon, J. M. Dobbie, C. L. Dolph, M. D. Donsker, J. L. Doob, C. H. Dowker, Y. N. Dowker, M. J. Dresher, Roy Dubisch, W. L. Duren, Aryeh Dvoretzky, J. M. Earl, E. D. Eaves, Paul Eberhart, W. F. Eberlein, B. J. Eisenstadt, E. S. Elyash, Paul Erdös, G. C. Evans, H. P. Evans, Howard Eves, G. M. Ewing, A. B. Farnell, William Feller, H. H. Ferns, C. H. Fischer, L. R. Ford, W. C. Foreman, J. S. Frame, T. C. Fry, L. E. Fuller, R. E. Fullerton, M. G. Gaba, H. M. Gehman, F. C. Gentry, H. H. Germond, R. E. Gilman, Wallace Givens, A. M. Gleason, Michael Golomb, D. B. Goodner, R. N. Goss, W. H. Gottschalk, Cornelius Gouwens, R. E. Graves, J. W. Green, R. E. Greenwood, Edison Greer, F. L. Griffin, V. G. Grove, William Gustin, B. L. Hagen, D. T. Haimo, Franklin Haimo, Edwin Halfar, P. R. Halmos, O. H. Hamilton, K. E. Hazard, E. A. Hazlewood, G. A. Hedlund, E. D. Hellinger, Erik Hemmingsen, A. S. Henriques, G. A. Herr, J. G. Herriot, Fritz Herzog, M. R. Hestenes, Edwin Hewitt, T. H. Hildebrandt, J. J. L. Hinrichsen, G. P. Hochschild, J. L. Hodges, P. G. Hoel, D. L. Holl, Carl Holtom, Eberhard Hopf, R. E. Horton, L. A. Hostinsky, Harold Hotelling, E. M. Hove, W. A. Howard, H. K. Hughes, M. G. Humphreys, N. C. Hunsaker, C. C. Hurd, Witold Hurewicz, W. R. Hutcherson, C. A. Hutchinson, Børge Jessen, B. W. Jones, F. B. Jones, Shizuo Kakutani, G. K. Kalisch, L. H. Kanter, Irving Kaplansky, William Karush, Leo Katz, M. W. Keller, J. L. Kelley, Claribel Kendall, G. S. Ketchum, P. W. Ketchum, D. E. Kibbey, R. S. Kingsbury, J. R. Kline, H. S. Konijn,

H. L. Krall, M. S. Kramer, M. P. Lackey, O. E. Lancaster, Joseph Landin, R. E. Lane, R. E. Langer, H. D. Larsen, D. H. Leavens, W. G. Leavitt, J. R. Lee, Solomon Lefschetz, D. H. Lehmer, W. T. Lenser, W. J. LeVeque, Harry Levy, F. A. Lewis, M. T. Lewis, C. B. Lindquist, W. S. Loud, R. G. Lubben, R. B. McClenon, Dorothy McCoy, N. H. McCoy, L. H. McFarlan, E. J. McShane, G. W. Mackey, Saunders MacLane, Wilhelm Magnus, H. B. Mann, Morris Marden, Anna Marm, George Marsaglia, M. H. Martin, F. J. Massey, M. S. Matchett, J. R. Mayor, A. E. Meder, A. S. Merrill, B. C. Meyer, H. L. Meyer, R. R. Middlemiss, R. A. Miller, T. W. Moore, W. K. Moore, C. B. Morrey, F. R. Morris, D. R. Morrison, D. J. Morrow, S. B. Myers, W. M. Myers, Leopoldo Nachbin, E. D. Nering, J. D. Newburgh, August Newlander, C. V. Newsom, Jerzy Neyman, O. M. Nikodým, Ivan Niven, J. I. Northam, D. A. Norton, A. B. J. Novikoff, E. G. Olds, E. J. Olson, A. M. Ostrowski, J. C. Oxtoby, O. O. Pardee, L. E. Payne, Sam Perlis, R. P. Peterson, T. S. Peterson, M. M. Pihlblad, George Piranian, J. C. Polley, G. B. Price, R. C. Prim, A. L. Putnam, E. D. Rainville, J. F. Randolph, R. B. Rasmusen, L. T. Ratner, L. L. Rauch, M. O. Reade, O. H. Rechard, O. W. Rechard, M. S. Rees, P. K. Rees, W. T. Reid, Irving Reiner, I. M. Reiner, J. G. Renno, C. N. Reynolds, C. E. Rickart, F. A. Rickey, P. R. Rider, F. D. Rigby, L. G. Riggs, L. A. Ringenberg, E. K. Ritter, J. B. Robinson, R. M. Robinson, Arthur Rosenthal, J. B. Rosser, M. F. Rosskopf, Herman Rubin, W. A. Rutledge, Rafael Sanchez-Diaz, R. G. Sanger, L. J. Savage, A. C. Schaeffer, H. M. Schaerf, E. R. Schneckenburger, Lowell Schoenfeld, K. C. Schraut, Henry Schutzberger, I. E. Segal, George Seifert, B. R. Seth, A. S. Shapiro, H. C. Schaub, L. W. Sheridan, Jack Sherman, Seymour Sherman, Edward Silverman, Annette Sinclair, M. F. Smiley, A. H. Smith, F. C. Smith, G. W. Smith, H. L. Smith, S. S. Smith, H. K. Sohl, E. S. Sokolnikoff, I. S. Sokolnikoff, T. H. Southard, E. J. Specht, D. E. Spencer, V. E. Spencer, C. E. Springer, K. H. Stahl, B. M. Stewart, R. W. Stokes, E. B. Stouffer, D. M. Studley, Otto Szasz, A. H. Tappan, Alfred Tarski, A. H. Taub, Eugene Taylor, H. P. Thielman, J. M. Thomas, F. B. Thompson, R. M. Thrall, V. H. Tingey, Leonard Tornheim, H. M. Trent, A. W. Tucker, J. W. Tukey, A. R. Turquette, S. M. Ulam, J. L. Ullman, Gilbert Ulmer, W. R. Utz, H. S. Vandiver, A. H. Van Tuyl, V. J. Varineau, G. L. Walker, R. J. Walker, J. L. Walsh, S. E. Warschawski, M. A. Weber, Alexander Weinstein, P. A. White, G. W. Whitehead, A. L. Whiteman, W. F. Whitmore, Hassler Whitney, L. R. Wilcox, F. B. Wiley, S. S. Wilks, R. L. Wilson, G. M. Wing, R. M. Winger, Clement Winston, H. E. Wolfe, J. H. Wolfe, C. R. Wylie, J. L. Yarnell, L. C. Young, P. M. Young.

The Colloquium Lectures, on *Topological dynamics*, were presented by Professor G. A. Hedlund of Yale University on Tuesday afternoon and Wednesday, Thursday, and Friday mornings. Presiding were, in order, President J. L. Walsh, Vice President Hassler Whitney, Dean W. L. Ayers, and Professor W. H. Gottschalk.

On Wednesday morning an invited address, *Some recent investigations in almost periodic functions*, was delivered by Professor Børge Jessen of the University of Copenhagen and the Institute for Advanced Study. Professor E. F. Beckenbach presided. A second invited address, *Aposyndetic and non-aposyndetic continua*, was delivered Thursday afternoon by Professor F. B. Jones of the University of Texas. Professor S. B. Myers presided at this meeting.

President Walsh presided at the business meeting on Wednesday morning. It was announced that Dean R. G. D. Richardson had died during the summer, and those present stand a moment in respect for his memory. Those presiding at the sectional sessions were as follows: Professors S. E. Warschawski, A. H. Taub, N. H. McCoy, P. R. Rider (Joint session with I.M.S.), H. F. Bohnenblust, J. L. Kelley, J. M. Thomas, Reinhold Baer, R. M. Winger, J. W. Calkin, B. W. Jones, and A. H. Diamond.

Headquarters for the meeting were in the Third Dormitory. The extensive facilities of this and adjacent dormitories were open to those attending the meetings.

On Tuesday afternoon there was a tea at the home of President Stearns. Wednesday afternoon was devoted to an excursion to Rocky Mountain National Park. On Thursday evening a steak fry was held on Flagstaff Mountain. On this occasion a resolution was presented by Dr. C. V. Newsom on behalf of the four participating societies, thanking the University of Colorado, the members of the Department of Mathematics, and all those who had contributed to the arrangements and extended their hospitality.

The Council met at 8:00 P.M. on August 30 and at 10:00 A.M. on September 1.

The Secretary announced the election of the following fifty-one persons to ordinary membership in the Society:

Dr. Harry Igor Ansoff, Rand Corporation, 1500 4th St., Santa Monica, Calif.;
Professor Margaret Malone Baskerville, Shorter College, Rome, Ga.;
Mr. Martin Roy Bates, Cornell University, Ithaca, N. Y.;
Professor Raymond Frank Bell, Eastern Washington College of Education, Cheney, Wash.;
Mr. Harold Dean Brown, Department of Physics and Astronomy, University of Kansas, Lawrence, Kan.;
Mr. Charles Carpenter Buck, University of Michigan, Ann Arbor, Mich.;
Professor Pedro Gerardo Cabezas, University of Cuyo, Cuyo, Argentina;
Mr. Sanford Colbert, 500 West 140th Street, New York, N. Y.;
Miss Helen Frances Cullen, University of Michigan, Ann Arbor, Mich.;
Mr. Thomas Scott Dean, 5930 Palo Pinto, Dallas, Tex.;
Mr. George William Evans, II, New York University, New York, N. Y.;
Mr. William Buell Evans, University of Illinois, Urbana, Ill.;
Mr. William Forman, Brooklyn College, Brooklyn 10, N. Y.;
Mr. Christopher Paul Gadsden, Tulane University of Louisiana, New Orleans, La.;
Mr. Robert Doran Glauz, University of Michigan, Ann Arbor, Mich.;
Mr. James Francis Hannan, University of North Carolina, Chapel Hill, N. C.;
Mr. Julius Honig, 6801 Bay Parkway, Brooklyn 4, N. Y.;
Mr. David Warren Hullinghorst, 2525 Peniston St., New Orleans 15, La.;
Mr. William Robert Hydeman, U. S. Navy, 3810 39th St., N.W., Washington 16, D. C.;
Professor Samuel Jacob Jasper, Kent State University, Kent, Ohio;
Mr. Paul Constantine Karpov, 634 West 135th Street, New York, N. Y.;

Mr. Harold William Kuhn, Princeton University, Princeton, N. J.;
Mr. Edward David Lawler, 268 Bowman Ave., Merion Station, Pa.;
Mr. Ernest John MacKenzie, Whitney Institute School, Bermuda;
Dr. Wilhelm Magnus, California Institute of Technology, Pasadena 4, Calif.;
Professor Esteban Francisco R. Manfredi, University of LaPlata, La Plata, B. A., Argentina;
Mr. Raymond William Moller, Catholic University of America, Washington, D. C.;
Mr. Frederick Joseph Morton, 1421 North Broad Street, Philadelphia 22, Pa.;
Professor Maria Laura Moura Mousinho, University of Brazil, Rio de Janeiro, Brazil;
Sister Maria Socorro Piccirillo, Immaculata College, Immaculata, Pa.;
Mr. Henry Polowy, Ferris Hall, North Brothers Island, New York, N. Y.;
Mr. Lyle Eugene Pursell, Purdue University, Lafayette, Ind.;
Mr. Edgar Reich, Rand Corporation, Santa Monica, Calif.;
Dr. Murray Rosenblatt, Cornell University, Ithaca, N. Y.;
Miss Shirley Anne Rubenstein, University of Virginia, Charlottesville, Va.;
Mr. William Trumbo Sandlin, Box 1606, Huntington, W. Va.;
Mr. Joseph Arthur Schatz, Brown University, Providence 12, R. I.;
Dr. Francis James Scheid, Boston University, Boston 16, Mass.;
Mr. David Harry Shaftman, 1156½ E. 61st St., Chicago 37, Ill.;
Mr. Stanley Musgrave Shartle, Office of Surveyor of Hendricks County, Danville, Ind.;
Mr. Daniel Shanks, Naval Ordnance Laboratory, White Oak, Silver Spring, Md.;
Mr. Frank R. Simpson, Franklin Institute, Philadelphia, Pa.;
Mr. Irving Sklansky, City College, New York 31, N. Y.;
Mr. Thomas Joseph Smith, College of the Holy Cross, Worcester, Mass.;
Major Mariano del Rosario Talag, Philippine Military Academy, Baguio, Philippines;
Mr. Nathaniel Thompson, Seton Hall College, South Orange, N. J.;
Mr. Robert Wasserman, University of Michigan, Ann Arbor, Mich.;
Mr. William Swoll Winn, Millsaps College, Jackson, Miss.;
Mr. Otto Wolf, 2 Hinckley Pl., Brooklyn 18, N. Y.;
Mr. David Monaghan Young, Harvard University, Cambridge 38, Mass.;
Mr. Neal Zierler, 2318 Eutaw Place, Baltimore, Md.

It was reported that the following person had been elected to membership on nomination of the University of Toronto: Mr. Gerald Berman, teaching fellow.

The Secretary announced that the following had been admitted to the Society in accordance with reciprocity agreements with other mathematical organizations: London Mathematical Society: Mr. Abram Samoilovitch Besicovitch, Trinity College, Cambridge University; Mr. Arthur ap Gywnn, University College of Wales; Dr. Kurt Angustus Hirsch, King's College, University of Durham; Swiss Mathematical Society: Professor Felix Fiala, University of Neuchâtel.

The following appointments by President J. L. Walsh of representatives of the Society were reported: Professor Eric Reissner on the U. S. Committee on Theoretical and Applied Mechanics for a term ending December 31, 1952; Professor Hassler Whitney at Ceremonial Session of Semi-Centennial Meeting of American Physical

Society on June 16, 1949; Professor F. A. Lewis at the inauguration of Ralph Brown Draughon as President of Alabama Polytechnic Institute on May 12, 1949; Professor B. W. Jones at the Seventy-fifth Anniversay of the Colorado School of Mines on September 30–October 1, 1949; Professor C. B. Morrey, Jr., at the inauguration of Dr. J. E. Wallace Sterling as President of Stanford University on October 7, 1949.

The following additional appointments by the President were reported: Professor R. V. Churchill as a member of the Editorial Committee for Applied Mathematics Symposium Proceedings for three years beginning July 1, 1949 (Professor Churchill appointed by committee to serve as Chairman for one year beginning July 1, 1949); Professors A. W. Tucker (Chairman), G. T. Whyburn, and J. W. T. Youngs as a committee to consider the non-editorial problems in connection with the Memoirs; Professors Saunders MacLane (Chairman), J. L. Doob, Dean A. E. Meder, Jr., Professors G. B. Price and A. W. Tucker as a committee to consider institutional membership rates, terms of individual membership, and the relation of the Society to institutional members which publish mathematical journals.

It was reported that the Society had signed a contract with the Office of Air Research whereby a subsidy of $21,500 will be available for the support of Mathematical Reviews for a period of one year.

It was reported that an arrangement had been made with the McGraw-Hill Book Company for the publication of the Proceedings of the Third Symposium in Applied Mathematics.

The Council voted to recommend to the Board of Trustees its approval of an arrangement between the Society and the McGraw-Hill Book Company for providing a limited supply of reprints to authors of articles in the Proceedings of the Third Symposium in Applied Mathematics.

A reciprocity agreement has been established between the Society and the Polish Mathematical Society, the terms being similar to those of other reciprocity agreements.

The following resolution on the death of Dean R. G. D. Richardson was adopted by the Council:

A long record of most distinguished service to the Society came to its close with the death, on July 17, 1949, of Roland George Dwight Richardson. Although for some years his health had been uncertain, the end came suddenly, on a trip to his native province of Nova Scotia.

The first twelve years of Dean Richardson's career, after he received his doctor's degree at Yale in 1906, showed a very considerable ability in original investigation. His impetus was largely given by James Pierpont. He spent the year 1908–1909 at Göttingen where he was greatly influenced by Hilbert. Later he owed much to his association with G. D. Birkhoff. His papers, concerned mostly with integral equations, calculus of variations, and boundary value problems, appeared in various American and foreign publications. Meantime, in 1907, he had begun his lifelong service as Professor, and later as Dean, at Brown University.

In 1915 he became Chairman of the Department of Mathematics at Brown, and turned his energies toward the upbuilding of that department to its present high position. If this had been his only service to American mathematics, it would have been a notable one. In his later years he had a leading part in making Brown a center of advanced instruction in applied mathematics. He also served for over twenty years as Dean of the Graduate School of Brown University. On his retirement in 1948 he was awarded the honorary degree of LL.D. by Brown University. Previously he had received honorary degrees from Acadia University and Lehigh University. For a year and a half before his death, he was Vice President of the American Academy of Arts and Sciences.

This burden of administrative work for his own university makes all the more impressive his devotion to the affairs of the Society. In 1920 reorganization of the offices of the Society became urgent with the retirement of Professor F. N. Cole who, with failing health, had insisted on personally attending to all clerical details. Dean Richardson was a Vice President at that time and was familiar with the business of the Society. When asked to become Secretary, he consented, though fearing that this would definitely end his career as a producing mathematician. He believed, however, that if his plans for the Society could be realized, the sacrifice would be more than worthwhile.

His unequalled services to the Society were fittingly described in a resolution adopted by the Council at the time of his retirement from the office of Secretary in 1940, as published in the Bulletin of March, 1941. This should be supplemented, however, by at least a mention of his membership on the Board of Trustees continuously since 1923.

His administration was never static. He always had a plan, well worked out, for new situations, yet never was over-insistent on his own views, and would listen courteously to others who were advancing perhaps less well-thought ideas. His sense of order and efficiency contributed to make the offices of the Society a model to other scientific organizations.

But with all due regard to what he did for this Society and his university, the highest tribute is expressed in this sentence of the 1940 resolution: "Richardson's real memorial lies, however, in the affection and admiration for him which are widespread among the present members of the Society, and universal among those who have known him personally and worked with him."

The following dates of meetings in 1950 were approved: February 25 in New York City; February 24–25 in East Lansing, Michigan; April 21–22 in Oak Ridge, Tennessee; April 28–29 in Washington, D.C., April 28–29 in Chicago, Illinois; April 29 in Berkeley, California; June 16–17 in Seattle, Washington.

Certain invitations to deliver hour addresses were announced: Professor R. H. Fox for the October, 1949, meeting in New York City; Professor Richard Bellman for the November, 1949, meeting in Pasadena; Professor R. H. Bruck for the November, 1949, meeting in

Chicago; Professors L. V. Ahlfors and S. Chowla for the 1949 Annual Meeting.

Professor G. A. Hedlund was elected (subject to confirmation by the Board of Trustees) as a member of the Bulletin Editorial Committee, to serve for the unexpired term of Professor Deane Montgomery, resigned (that is, until December 31, 1951).

Professors H. F. Bohnenblust and L. R. Ford were elected representatives of the Society on the Council of the American Association for the Advancement of Science for the years 1950 and 1951.

Professors G. C. Evans and Einar Hille were nominated as representatives of the Society in the Division of Mathematical and Physical Sciences of the National Research Council for a three-year period beginning July 1, 1950.

Professor C. B. Allendoerfer and Dr. R. P. Boas were elected representatives of the Society on the Editorial Board of the Duke Mathematical Journal for a period of three years beginning January 1, 1950.

The Council accepted the recommendation of the Committee to Nominate an Executive Director that Dr. H. M. MacNeille, now with the Atomic Energy Commission, be appointed to this position, subject to the approval of the Board of Trustees.

The Council accepted, with certain amendments, a report of the Committee on the Role of the Society in Mathematical Publication dealing with the problem of what constitutes publication.

Recommendations in connection with institutional membership dues were accepted by the Council as follows: that the page rate for institutional dues be raised to $5.00 per page (present rate is $2.75 per page); that dues be based on averages computed triennially over three-year periods, with certain provisions for the transition period; that the page counts for institutional dues cover the American Journal, Annals, Bulletin, Transactions, Proceedings (when established) and the Canadian Journal; that the privileges for institutional members be revised according to a table determined by the Council and approved by the Trustees; that the new scale for institutional members go into effect beginning January 1, 1951.

It was reported that the Memoirs would be inaugurated in the near future, as soon as arrangements could be made for the preparation of manuscripts already accepted.

The Council voted to invite Professor G. E. Uhlenbeck of the University of Michigan to deliver the Josiah Willard Gibbs Lecture in 1950.

The Committee on Revisions of By-Laws recommended certain changes in connection with (1) the time of the annual meeting of the Board of Trustees; (2) the method of arranging for temporary replacements on Editorial Committees; (3) the inauguration of the Proceedings and Memoirs. These changes will be acted upon at the October meeting of the Society in New York and will be effective January 1, 1950.

The Council voted to abolish the Committee on Publicity; this action was taken at the request of the committee and it was the sense of the meeting that the activities of this committee would now be carried on by the Executive Director.

The Council voted to adopt the following resolution in connection with the loyalty oath section of the National Science Foundation Bill being considered by the House of Representatives:

> The Council of the American Mathematical Society expresses its hearty approval of the National Science Foundation Bill (S247) as passed by the United States Senate, and confidently expects that an early passage of a similar bill by the House of Representatives will be a potent factor in the development of science in this country.
>
> With respect to the amendments introduced into the Bill by the House committee, the Council deplores the amendment inserting a new section, Section 10(b), which introduces non-scientific objectives.

Abstracts of papers presented at the meeting follow below. Papers whose titles are followed by "t" were read by title. Some of the abstracts of papers read by title were published in September. Mr. Conrad and Mr. Goss were introduced by Professor D. L. Holl, Professor Buchi and Professor Magnus by Professor Saunders MacLane, Professor Spencer by Professor R. B. Lindsay, Professor Dresden and Mr. Brown by Professor G. B. Price, Mr. Calderon by Professor Zygmund, Mr. Lieblein, Dr. Castellani, and Professor Chang by Professor J. W. Green. Paper 515 was presented by Professor Herzog, paper 536 by Mr. Riggs, paper 568 in the absence of the author by Dr. H. M. Trent, paper 470 by Professor Perlis, paper 577 by Mrs. Szmielew, paper 586 by Dr. Sherman, paper 581 by Professor Dolph, paper 517 by Professor Hewitt, paper 523 by Dr. Farnell, paper 472 by Professor Tucker, paper 464 by Professor Barnett, paper 473 by Professor Givens, paper 531 by Professor Piranian, paper 462 by Professor Abbott, and paper 563 by Dr. Van Tuyl.

Algebra and Theory of Numbers

462. J. C. Abbott and T. J. Benac: *Right congruences on groupoids.*

A right congruence θ on a right quasi-group G is any equivalence preserved by the group $\overline{\mathfrak{R}}$ generated by the right multiplications R_y of G. If G contains a left unit, then every coset has the same power. If either the order or index of θ is finite, then any equivalence preserved by the R_y alone is a right congruence (cf. Albert, Trans. Amer. Math. Soc. vol. 54 (1943) Theorem 6, and Bates and Kiokemeister, Bull. Amer. Math. Soc. vol. 54 (1948) Theorem 4). The set of right congruences forms a complete lattice which is not modular since it does satisfy Dubreil's condition (Dubreil, *Algèbre*, 1946). If the kernel of the right congruence is called right normal, then a sub-right quasi-group will be right normal if and only if $H\mathfrak{J}_r = H$ where $\mathfrak{J}_r$ is the right inner mapping group of G (Bruck, Trans. Amer. Math. Soc. vol. 60 (1946)). $\mathfrak{J}_r$ contains no subgroup normal in $\overline{\mathfrak{R}}$, and the set of R_y's forms a complete set of representatives of G mod θ, so that $(\overline{\mathfrak{R}}/\mathfrak{J}_r;\ R_y)$ is a Baer canonical representation of G. Finally, every right congruence of G can be extended to a congruence on $\overline{\mathfrak{R}}$. (Received July 22, 1949.)

463. R. V. Andree: *The sequence development of g-adic rings.*

The usual definition of valuation is modified by replacing the equality in $\phi(xy) = \phi(x)\phi(y)$ by $\leqq$. The g-adic numbers (g prime or composite) are then obtained as sequences of rational numbers which converge with respect to this generalized valuation. It is proved that for every g containing at least two prime factors, and for every integer n, there exists an equation of degree n having no solution in Ω_g. It is established that every g-adic ring is isomorphic to a direct sum of those p-adic fields for which p divides g. Through this isomorphism a finite process is obtained for determining the exact number of distinct solutions a given g-adic equation will have, and a non-tentative method for producing every solution is developed. In conclusion it is proved that every g-adic ring is perfect (complete) with respect to the generalized valuation. (Received June 21, 1949.)

464. I. A. Barnett and C. W. Mendel: *Fermat's last theorem in matrices with integral elements.*

The equations $X^n + Y^n + Z^n = 0$ are considered where X, Y, Z are square matrices of the same order with integral or zero elements. If X and Y are commutative, the most general solution in rationals of the equation is found when $n = 2$. Solutions for any n in integral matrices of order n are exhibited. For $n = 2$, the general solution for 2×2 integral matrices is given. Finally, with the Fermat conjecture assumed to be true, we let X be a two-rowed matrix with first row x_1, x_2 and second row x_3, x_4, and so on, and $D \neq 0$, where D is the three-rowed determinant with first row $x_1 - x_4$, $y_1 - y_4$, $z_1 - z_4$, second row x_2, y_2, z_2, and third row x_3, y_3, z_3. Then the only solution of the equation in 2×2 integral matrices for n odd are those for which the traces of two of the matrices are numerically equal with the corresponding determinants equal, while the trace and the determinant of the third matrix are both zero. (Received July 14, 1949.)

465. P. T. Bateman: *On the distribution of kth-power-free integers.*

Known prime number theory implies that the number of kth-power-free integers not exceeding x is $x/\zeta(k) + O(x^{1/k} \exp\{-A(\log x)^\alpha\})$, where A is a positive constant and α is a constant between 1/2 and 1, in fact, the same constant that appears in the error term for the number of prime numbers not exceeding x. It is pointed out here that the Riemann hypothesis implies only that the above error term can be improved to $O(x^{2/(2k+1)+\epsilon})$. These error terms give upper estimates for the differences between consecutive kth-power-free integers. On the other side it is shown that the difference between a kth-power-free integer q and the next kth-power-free integer is greater than $(\log q)/k(\log \log q)$ infinitely often. (Received July 18, 1949.)

466. J. H. Bell: *The solution of a unilateral direct product matrix equation.*

The author's Bull. Amer. Math. Soc. Abstract 54-11-417 gives necessary and sufficient conditions for the solution of the matrix equation $\sum_{m=0}^{s} A_m \cdot xX^m = 0$ over a field $\mathcal{J}$. In this paper the equation is generalized to $\sum_{m=0}^{s} A_m \cdot x(K_m X^m) = 0$. It is shown that the solution of this generalized equation may be reduced to the solution of r simultaneous unilateral equations $\sum_{m=0}^{s} a_{m,ij} K_m X^m = 0$, where $A_m = (a_{m,ij})$ and r is the number of linearly independent matrices in the set $A_s, A_{s-1}, \cdots, A_0$. If the dimensions of A_m, K_m, and X are such that $\sum_{m=0}^{s} A_m \cdot x(K_m X^m)$ is a square matrix, an alternative method of solution, patterned after Ingraham's algorithm for the solution of the unilateral matrix equation, is given. In the development of this method a remainder theorem and a divisor theorem, involving direct products, are obtained. (Received June 10, 1949.)

467. A. W. Boldyreff: *Reduction of homogeneous diophantine equations to systems of linear homogeneous equations.*

This paper considers the possibility of reduction of homogeneous diophantine equations to systems of linear homogeneous equations, directly or with the aid of algebraic identities, and develops methods of deducing the solutions of the former from those of the latter. The methods are illustrated by applications to a number of well known classical problems, and it is shown how they can be modified to apply to the solution of non-homogeneous equations. (Received August 13, 1949.)

468. J. R. Buchi: *Boolean orderings and pairings.*

The paper has its origin in the idea of an axiomatic treatment of the inclusion relation of set theory. By taking inclusion as primitive rather than the relation "is element of," it becomes unnecessary to assume that there are atoms (as the elements in ordinary set theory). The best way to carry out such an axiomatic approach is to take the disjunction relation as a second primitive. Four simple axioms complete the axiomatic system. It is shown that inclusion in the above context is a special type of partial ordering called Boolean ordering. The importance of this Boolean ordering in the theory of Boolean algebras is illustrated. For example, every Boolean ordered set can uniquely be embedded in a smallest complete Boolean algebra (analogy to the embedding of a general partially ordered set in a complete lattice). In the second part of the paper a concept called pairing is introduced. In terms of this concept the ordinary notion of mappings of sets is generalized to mappings of Boolean orderings. In other words, a definition of the notion "mapping" is given in the case where the structors which are to be mapped are not sets and therefore do not have atoms. (Received July 18, 1949.)

469t. Leonard Carlitz: *Some properties of Bernoulli numbers of higher order.*

Among the results of this paper, the following may be cited. 1. Let $k < p-1$, $m \not\equiv 0, 1, \cdots, k-1 \pmod{p-1}$, then $B_m^{(k)}/m^{(k)}$ is integral (mod p), $m^{(k)} = m(m-1) \cdots (m-k+1)$. Also $\sum_{i=0}^{r} (-1)^i C_{r,i} T_{m+i(p-1)} \equiv 0 \pmod{p^r}$, where $T_m = B_m^{(k)}/m^{(k)}$. (The

$B_m^{(k)}$ notation is that of Nörlund.) 2. $B_p^{(p)} \equiv p^2/2 \pmod{p^3}$, $B_{p+2}^{(p+1)} \equiv 0 \pmod{p^3}$ $(p>3)$. The special case $B_{p+2}^{(p+1)} \equiv 0 \pmod{p^2}$ is due to S. Wachs, Bulletin des Sciences Mathématiques (2) vol. 71 (1947) pp. 219–232. (Received July 14, 1949.)

470. Roy Dubisch and Sam Perlis: *On total nilpotent algebras.*

A total nilpotent algebra $F^{(n)}$ of degree n over a field F is defined to be the totality of n-rowed square matrices over F with 0's on and above the main diagonal, or any isomorphic copy thereof. The importance of $F^{(n)}$ lies in the fact that every nilpotent associative algebra of order $n-1$ is isomorphic to a subalgebra of $F^{(n)}$, which itself is nilpotent. The authors determine all ideals of $F^{(n)}$, the group G of all automorphisms over F of $F^{(n)}$, some of the structural properties of G, all characteristic ideals C (ideals C such that $C^T=C$ for every T in G). Certain characteristic ideals D give rise to all others by the processes of intersection and join. The internal structural properties of these ideals D are investigated. (Received July 18, 1949.)

471. J. S. Frame: *A simple recursion formula for inverting a matrix.*

A simple recursion formula is obtained for computing the inverse of a matrix, and as a biproduct the characteristic equation, the λ-adjoint, and the characteristic vectors of the matrix. Let the $n\times n$ matrix A have the characteristic equation (1) $|\lambda I-A|=F(\lambda)=\lambda^n-\sum_1^n c_k\lambda^{n-k}=0$, let the adjoint of $\lambda I-A$ be denoted by (2) $C(\lambda)=A_0\lambda^{n-1}+A_1\lambda^{n-2}+\cdots+A_{n-1}$, so that (3) $(\lambda I-A)C(\lambda)=F(\lambda)I$, and define $A_n=0$. It is proved that we may compute successively $c_1, A_1, c_2, A_2, \cdots, c_{n-1}, A_{n-1}, c_n, A_n=0$ by means of the two formulas (4) $c_k=(1/k)$ trace (AA_{k-1}), (5) $A_k=AA_{k-1}-c_kI$. In particular the determinant and inverse are obtained as $|A|=(-1)^{n-1}c_n$, $A^{-1}=A_{n-1}/c_n$, if $c_n\neq 0$. Formula (4) is derived from (6) $F'(\lambda)=$ trace $C(\lambda)$, and (7) trace $AC(\lambda)=\lambda F'(\lambda)-nF(\lambda)$. The computation is finally checked if A_n comes out 0, but a method is given for correcting small errors without repeating the calculation. The relation $AA_k=A_kA$ provides a further check at each stage. The nonvanishing columns of $C(\lambda_k)$ are characteristic vectors if λ_k is a simple root of $F(\lambda)=0$. The characteristic vectors for multiple roots are found explicitly from the derivatives of $C(\lambda)$, and the results are applied to the solution of simultaneous linear differential equations with constant coefficients. (Received June 4, 1949.)

472. David Gale, Harold W. Kuhn, and A. W. Tucker: *Reductions of game matrices.*

Let a game matrix A of m rows and n columns be reduced to an $m-p+1$ by $n-q+1$ matrix A^* by combining linearly the first p rows of A into a single leading row and the first q columns into a single leading column, using as coefficients the components of a pair of optimal strategies for the game determined by the p by q submatrix common to the p rows and q columns. Two special cases of this reduction are noteworthy: (1) when the first p rows of A have the same last $n-q$ elements, and the first q columns have the same last $m-p$ elements; (2) when the coefficients are all positive, and the new leading element is a saddlepoint for A^*. In both cases A^* has the same game-value as A. In the first case a pair of optimal strategies for A^* unites with the two sets of coefficients to give a pair of optimal strategies for A. In the second case the two sets of (positive) coefficients themselves constitute a pair of optimal strategies for A. Results of Bohnenblust-Karlin-Shapley, Gale, and Sherman show that this reduction to a saddlepoint matrix is always possible, the maximal sets of p rows and q columns being unique. (Received July 18, 1949.)

473. Wallace Givens and W. A. Rutledge: *Symmtric and skew Hadamard matrices.* Preliminary report.

A square matrix H of order n is Hadamard if each of its elements is ± 1 and $H'H=nI_n$; n must be 2 or a multiple of 4 with existence for every such value being in doubt. If H is symmetric, it is shown that the signature is zero if n is not a square and is an even integer between $-n^{1/2}$ and $+n^{1/2}$ if n is a square. For various forms of n, symmetric H are constructed and for $n=2^{2k}$ all permissible signatures are obtained. When $H=I_n+S$ where S is skew symmetric, the invariant factors of S (over the ring of rational integers) are half $+1$ and half $(n-1)$ if $(n-1)$ is square free. If $(n-1)$ has a square factor they are restricted otherwise. The tensor set associated under the spinor representation of the orthogonal group with an Hadamard matrix of either of the above types has especially simple features. (Received July 18, 1949.)

474. L. Aileen Hostinsky: *Endomorphisms of lattices.*

The object of this paper is to develop a theory of homomorphisms for lattices by generalizing the concept of homomorphism as used in group theory. In addition to a derivation of the basic isomorphism theorems, a generalization of Fitting's lemma and of the theory of splitting endomorphisms is given. The main results are established by the introduction of the concept of an η-automorphic element. It is shown that a necessary condition for uniform splitting is that the sum of η-automorphic elements be η-automorphic and that under the assumption of uniform splitting the complement is uniquely determined as the sum of η-automorphic elements. Reduction theorems and certain necessary and sufficient conditions for uniform splitting are included. (Received July 5, 1949.)

475*t*. L. K. Hua: *A theorem on matrices over a sfield and its applications.*

Let Φ be a sfield (or a division ring). Suppose $1<n\leqq m$. Two matrices are called coherent if the rank of their difference is one. Any one-to-one mapping which carries $n\times m$ matrices into $n\times m$ matrices and leaves the coherence invariant is of the form $Z_1=PZ^{\sigma}Q+R$, where $P(=P^{(n)})$ and $Q(=Q^{(m)})$ are non-singular, and R is an $n\times m$ matrix and σ is an automorphism of the sfield Φ. When $m=n$, in addition to the previous one, we have also $Z_1=PZ'^{\sigma}Q+R$, where Z' is a transpose of Z and σ is an anti-automorphism of the sfield. There are quite a number of applications of the theorem. This paper contains three of them: (i) Grassmann geometry, (ii) Jordan ring, and (iii) Lie ring. For example, for the Jordan ring the author proves that any Jordan automorphism of a semi-simple ring with descending chain conditions for the left ideals is either an automorphism or an anti-automorphism. (Received July 12, 1949.)

476*t*. L. K. Hua: *On exponential sums over an algebraic number field.*

Let K be an algebraic field of degree n over the rational field, and d be the different of the field. Let $f(x)=\alpha_k x^k+\cdots+\alpha_1 x+\alpha_0$ be a polynomial of the kth degree with coefficients in the field K, and let a be the fractional ideal generated by $\alpha_k,\cdots,\alpha_1$. Let $ad=rq$, where r and q are two relatively prime integral ideals. Then the following result is proved: Let $S(f(x),q)=\sum e^{2\pi i\,\mathrm{tr}\,(f(x))}$, where x runs over a complete residue system, mod q. Then, for any given $\epsilon>0$, we have $S(f(x),q)=O(N(q)^{1-1/k+\epsilon})$, where the constant implied by the symbol O depends only on k, n, and ϵ. (Received July 12, 1949.)

477t. L. K. Hua: *On semi-automorphisms of a sfield.*

Let K be a sfield. A one-to-one mapping $x \to x^\sigma$ of K onto itself is called a semi-automorphism if $(a+b)^\sigma = a^\sigma + b^\sigma$ and $(aba)^\sigma = a^\sigma b^\sigma a^\sigma$. It is proved in the paper that any semi-automorphism is either an automorphism or an anti-automorphism. This settles a problem of Ancochea. As a consequence we solve the fundamental theorem of the projective geometry over a sfield. (Received July 12, 1949.)

478t. L. K. Hua: *On the automorphisms of classical groups.*

In this paper the author solves several problems which were left open by Dieudonné (C. R. Acad. Sci. Paris vol. 225 (1947) pp. 914–915) about the automorphisms of the classical groups. The groups under consideration are $GL_2(K)$, $PGL_2(K)$, $SL_4(K)$, and $PSL_4(K)$ for K any sfield, and $O_4^+(K)$ and $PO_4^+(K)$ for K any field. The situation concerning $GL_2(K)$ and its related groups is unexpectedly successful, since there is no obscure notion like semi-automorphism in the final result. Concerning $O_4^+(K)$ we found some new automorphisms which were not known before. (Received July 12, 1949.)

479t. L. K. Hua: *Some properties of a sfield.*

The results of the paper are initiated from an almost trivial identity: if $ab \neq ba$, then $a = (b^{-1} - (a-1)^{-1}b^{-1}(a-1))(a^{-1}ba - (a-1)^{-1}b^{-1}(a-1))^{-1}$. Among the results, there are two interesting ones which are the perfect form of two theorems due to H. Cartan and J. Dieudonné respectively. (i) Every sfield is generated by a non-central element and its conjugate. (ii) The projective special linear group over the sfield K is simple except when $n=2$ and K has 2 or 3 elements. It is easily proved that both the upper central series and the lower central series of the multiplicative group of a sfield do not exist. The author proved also that the multiplicative group of a sfield is either abelian or non-metabelian. (Received July 12, 1949.)

480t. Nathan Jacobson and C. E. Rickart: *Jordan homomorphisms of rings.*

A mapping $a \to a^J$ of an arbitrary ring A into a second ring B is called a Jordan homomorphism provided $(a+b)^J = a^J + b^J$, $(a^2)^J = (a^J)^2$, $(aba)^J = a^J b^J a^J$. If B is an integral domain, then $a \to a^J$ is either a ring homomorphism or an anti-homomorphism. The proof of this is elementary and is an extension of a proof given by L. K. Hua (unpublished) for the case of a Jordan automorphism of a division ring. If A and B are primitive rings with minimal ideals and $a \to a^J$ is one-to-one *onto* B, then $a \to a^J$ is either a ring isomorphism or an anti-isomorphism. If A contains a system of matrix units $e_{ij}(i, j=1, \cdots, n;\ n \geqq 2)$ and a subring C whose elements commute with the e_{ij} and such that every element of A is of the form $\sum c_{ij}e_{ij}$, $c_{ij} \in C$, then there exists a direct sum decomposition $E_J = E' + E''$ of the enveloping ring E_J of A^J such that $a^J = a^{J'} + a^{J''}$, where $a \to a^{J'}$ is a ring homormophism onto E' and $a \to a^{J''}$ is a ring anti-homomorphism onto E''. The proof here is suggested in part by methods used previously by F. D. Jacobson and N. Jacobson [Trans. Amer. Math. Soc. vol. 65 (1949) pp. 141–169]. (Received July 18, 1949.)

481. D. H. Lehmer: *On the distribution of totitives.*

This paper is concerned with the distribution of the numbers not exceeding n and prime to n, whose number is Euler's function $\phi(n)$. Following Sylvester, these numbers are called the totitives of n. The function $\phi(k, q, n)$ is defined (for integer q, k, n, such that $0 \leqq q < k$) as the number of totitives of n lying between nq/k exclusive, and $n(q+1)/k$ inclusive. To measure the departure from uniformity in the distribution of the totitives we define the function $E(k, q, n) = \phi(n) - k\phi(k, q, n)$. Properties of $E(k, q, n)$ are studied. This function can be evaluated explicitly for $k = 1, 2, 3, 4$, and 6 in terms of the Liouville function $\lambda(n)$ and the Euler function $\theta(n) = 2^t$, where t is the number of distinct primes dividing n. Special results hold for other values of k. $E(k, q, n)$ is uniformly small in q. Its mean value for $n \leqq x$ is, in absolute value, less than $k \log x$ for every $q < k$. Applications are given to certain problems in cyclotomy. (Received July 18, 1949.)

482*t*. Benjamin Lepson: *Some examples related to the $\alpha+\beta$ hypothesis in the theory of Schnirelmann density.*

The $\alpha+\beta$ hypothesis, first proved by Mann (Ann. of Math. vol. 43 (1942) pp. 523–527), states that the density γ of the sum of two sets, each containing zero, and of densities α and β respectively, is at least min $(1, \alpha+\beta)$. It is shown in the present paper that, for any two non-negative real numbers α and β with $\alpha+\beta \leqq 1$, the g.l.b. of the numbers γ, each of which is the density of a set which is the sum of a set of density α containing zero and a set of density β containing zero, is $\alpha+\beta$. If α and β are both rational, there is a set as above whose density is equal to $\alpha+\beta$. This proves that the $\alpha+\beta$ hypothesis is best possible in the strong sense that the function $\alpha+\beta$ cannot be replaced by another function $f(\alpha, \beta)$ with $f(\alpha, \beta) > \alpha+\beta$ for some pair (α, β). For the rational case, the construction involves congruence classes of integers. The general case depends upon the lemma that any set A of density α contains a subset B of density α' where α and α' are any real numbers satisfying the inequality $0 \leqq \alpha' < \alpha \leqq 1$. (Received July 18, 1949.)

483. Wilhelm Magnus: *On the Baker-Hausdorff formula.*

If $\exp x \exp y = \exp z$ where x, y, are noncommutative variables, then z is a sum of alternantes or Lie-elements of x, y with rational coefficients. The terms whose coefficients have denominators which are divisible by p (a fixed prime number) are shown to lie (possibly with a finite number of exceptions) in an ideal which can be derived from a single element by means of algebraic substitutions and Baker-Hausdorff differentiation. From this a method can be derived to construct groups for which the pth power of every element equals the unit element and which are of a class greater than p. (Received July 18, 1949.)

484. D. A. Norton: *Hamiltonian loops.*

A loop G is Hamiltonian if every subloop is normal. It is power-associative if $\{x\}$ is a group for every $x \in G$. If G is the direct product of Hamiltonian loops M, N, it is Hamiltonian if and only if for each pair H, K of subloops of M, N respectively which contain subloops H', K' such that $H/H' \cong K/K'$, then H/H' and K/K' are in the centers of M/H' and N/K' respectively. Given any series of integers excluding 4, there exists a loop which contains only one series of composition, with these as factors of composition, and which contains no other proper subloops. A finite Hamiltonian power-associative loop is the direct product of prime-power order loops. A finite Hamiltonian Moufang loop is a direct product of an Abelian group of odd order, an Abelian group of order 2^m, type $[2, 2, \cdots, 2]$, and a loop of order 2^n of which the center is the two-group, each two independent elements generate a quaternian group, and each three generate a Cayley loop (the multiplicative loop of the basis elements of the Cayley-Dickson algebra). (Received July 14, 1949.)

485*t*. K. G. Ramanathan: *The theory of units of quadratic and hermitian forms.*

Let K be a totally real algebraic number field of finite degree $f>1$ over the field of rationals. Let S be the matrix of a nondegenerate quadratic form with m variables and with elements integral in K. Let n_r, $m-n_r$ $(r=1, \cdots, f)$ be the signatures of the conjugates of S. Let Γ be the group of units u of S in which u and $-u$ are identified. The following results are proved. (1) Γ is finite if and only if S is total-definitive. (2) Γ is finitely generated. (3) There is a space H of $\sum_{r=1}^{f} n_r(m-n_r)$ dimensions in which Γ has a faithful and discontinuous representation as a group G of non-euclidean motions leaving H invariant. (4) In H there exists a fundamental domain F in regard to G bounded by a finite number of hyperplanes. (5) In H can be defined an invariant volume element measured with which F has a finite volume V. (6) V, apart from a constant factor, is the reciprocal of the order of Γ if it is finite. Exactly similar results are proved for the case when S is an Hermitian matrix with integral elements in $K(d^{1/2})$ where d is a totally-negative integer of K. (Received July 18, 1949.)

486*t*. W. T. Reid: *A note on the characteristic polynomials of certain matrices.*

This note establishes the following results for matrices with elements in an arbitrary field: If A is an $n\times m$ matrix and D is an $n\times n$ matrix, then $|\lambda I-AB|=|\lambda I-AB-D|$ for arbitrary $m\times n$ matrices B if and only if D is nilpotent and $DA=0$. In particular, it follows that if C is an $m\times n$ matrix then the condition $ACA=0$ is necessary and sufficient for $|\lambda I-AB|=|\lambda I-A(B+C)|$ for arbitrary $m\times n$ matrices B; the sufficiency of this condition has been obtained recently by W. V. Parker [Bull. Amer. Math. Soc. vol. 55 (1949) pp. 115–116]. (Received July 18, 1949.)

487*t*. C. E. Rickart: *Isomorphism of groups of linear transformations. II.*

Consider a system $(\mathfrak{X}, \mathcal{D}, \mathfrak{X}^*: \mathcal{G})$ in which $\mathfrak{X}$, $\mathfrak{X}^*$ are respectively right and left linear vector spaces over the division ring $\mathcal{D}$, which are dual in the sense of Jacobson [Ann. of Math. vol. 48 (1947) pp. 8–21] with dimension greater than two, and $\mathcal{G}$ is a group of linear transformations on $\mathfrak{X}$ under the operation $A\circ B=A+B-AB$. Assume further that every $A\in\mathcal{G}$ possesses an adjoint (loc. cit., p. 16) and that $\mathcal{G}$ contains every one-dimensional involution which possesses an adjoint (T is an involution if $T\circ T=0$). Let $(\mathfrak{Y}, \mathcal{E}, \mathfrak{Y}^*: \mathcal{H})$ be a second such system and assume $\mathcal{G}$ and $\mathcal{H}$ isomorphic as groups under the mapping $G\rightarrow g(G)$. It is proved here that there exists a homomorphism $G\rightarrow G^{\epsilon}$ of $\mathcal{G}$ into the center of $\mathcal{E}$ (under the operation $\alpha\circ\beta=\alpha+\beta-\alpha\beta$) such that $g(G)$ either has the form $g(G)=(\phi^{-1}G\phi)\circ IG^{\epsilon}$, where $x\rightarrow x\phi$ is an isomorphism of the space $\mathfrak{X}$ onto the space $\mathfrak{Y}$, or $g(G)^*$ has an analogous form involving the inverse of G and an anti-isomorphism $x\rightarrow x\phi^*$ of $\mathfrak{X}$ onto $\mathfrak{Y}^*$. This result essentially includes a similar result of Dieudonné for the finite-dimensional case [C. R. Acad. Sci. Paris vol. 225 (1947) pp. 914–915] and uses a result previously announced by the author [Amer. Math. Soc. Abstract 54-11-440]. (Received July 18, 1949.)

488*t*. C. E. Rickart: *Representation of certain Banach algebras on Hilbert space.*

Let A denote a real Banach algebra which contains a minimal ideal R and in which there is defined a mapping $x \rightarrow x^*$ satisfying the following three conditions: (1) $(x^*)^* = x$, (2) $(xy)^* = y^*x^*$, (3) $xx^* = 0$ implies $x = 0$. Then there exists an idempotent $e \in R$ such that $e^* = e$, $R = eA$, and eAe is either the reals, quaternions, or complexes. Let R have dimension greater than one, greater than four, or infinite according as eAe is the reals, quaternions, or complexes. Then a real inner product (x, y) can be introduced into R such that $(xa, y) = (x, ya^*)$ for all $x, y \in R$ and $a \in A$. In general, additional conditions are needed to ensure that the norm $|x| = (x, x)^{1/2}$ be equivalent to the given norm in R. The results here include those of Kakutani and Mackey [Ann. of Math. vol. 45 (1944) pp. 50–58; Bull. Amer. Math. Soc. vol. 52 (1946) pp. 727–733] on the characterization of Hilbert space in terms of its ring of bounded operators. (Received July 18, 1949.)

489. R. M. Robinson: *Undecidable rings.*

The solution to the decision problem is shown to be negative for the ring of algebraic integers in a quadratic field, for a polynomial ring over a field, and for various other rings. These results follow from the fact that all problems of the arithmetic of natural numbers can be stated in the arithmetical theory of each of the rings mentioned. (Received June 27, 1949.)

490. J. B. Rosser: *Real roots of Dirichlet L-series.*

Let k be a positive integer and take any real character (mod k) and let L be the corresponding Dirichlet L-series (see Landau, *Vorlesungen über Zahlentheorie*, vol. 1, pp. 83–88). In the present paper it is proved that for $2 \leqq k \leqq 67$, L has no positive real zeros. (Received July 5, 1949.)

491*t*. W. E. Roth: *On the equations $AX - YB = C$ and $AX - XB = C$ in matrices.*

The necessary and sufficient condition that the equation $AX - YB = C$, where A, B, and C are rectangular matrices with elements in any field F, have a solution is that the 2-rowed matrix with first row A, C and second row 0, B and the two-rowed matrix with first row A, 0 and second row 0, B be equivalent. In case $X = Y$ and the matrices A, B, and C are square, the condition is that these matrices be similar. (Received July 14, 1949.)

492*t*. W. E. Roth: *On the matric equation $X^2 + AX + XB + C = 0$.*

The equation $X^2 + AX + XB + C = 0$, where A, B, and C are $n \times n$ matrices with elements in F, the field of complex numbers or a subfield thereof, is solved for $n \times n$ matrices X with elements in F. The methods are entirely rational under the operations of F. Though the equation becomes unilateral if X be replaced by $Y - A$ or by $Z - B$, this fact is not utilized for it was noted by the author only after the results of the present paper had been discovered. (Received July 5, 1949.)

493. Herman Rubin: *Postulates for the existence of measurable utility and pyschological probability.*

Von Neumann and Morgenstern in *Theory of games and economic behavior* show that under certain axioms a rational man will maximize a function, called the *utility* *function*, of probability distributions of future histories of nature. This function is unique except for a monotonic linear transformation. The present paper considers, instead of distributions of future states of nature, prospects, that is, functions from the present state to them. It is assumed that prospects are partially ordered, that given any two prospects, a rational man can select a unique random combination of them which he prefers, that if he cannot influence the consequences of a random event E, his choice will be independent of E, and that if a prospect is between two others, it is equivalent to a random combination of them. These postulates yield a utility function. If it is further assumed that every value of a prospect is the value of a constant prospect, then we can map the prospects linearly into real-valued functions on the states of nature. If these functions form a lattice, and utility of a prospect is monotone, then a rational man will act as if he believes in psychological probability. (Received July 16, 1949.)

494*t*. Peter Scherk: *Decomposition of orthogonalities into symmetries.*

Let A, G, T denote $n\times n$ matrices over a field of characteristic $\neq 2$. The fixed matrix G is assumed to be regular and symmetric. T is called orthogonal if $T'GT=G$. If A is orthogonal and rank $(A-I)=1$, A is called a symmetry [I=unit matrix]. From a theorem by Cartan every orthogonality can be decomposed into a product of n or less than n symmetries [cf. Dieudonné, *Sur les groupes classiques*, Actualités Scientifiques et Industrelles, no. 1040, 1948, pp. 20–22]. Let m=rank $(T-I)$. The author proves: If $G(T-I)$ is not skew-symmetric, then T can be written as a product of m but not less than m symmetries. If $G(T-I)$ is skew-symmetric, then $m\leqq n/2$—and, of course, $m\equiv 0 \pmod 2$—and this minimum number is equal to $m+2$. This exceptional case is studied in detail. (Received July 11, 1949.)

495. M. F. Smiley: *Special alternative rings.*

A study, initiated in a previous paper (Ann. of Math. (2) vol. 49 (1948) pp. 702–709), of the possibility of extending Jacobson's structure theory of associative rings (Amer. J. Math. vol. 67 (1945) pp. 300–320) to alternative rings is continued. An alternative ring A is called *special* if for every a, b, $c\in A$, the equation $a(bc)-(ab)c=x+ax$ has a solution $x\in A$. It is proved that *a special alternative ring has zero Jacobson radical if and only if it is a subdirect sum of special primitive alternative rings.* As a prelininary result it is shown that the Jacobson radical of a special alternative ring is the intersection of the family of regular maximal right ideals (cf. I. Kaplansky, Ann. of Math. (2) vol. 49 (1948) p. 692). Special alternative rings include all direct sums whose summands are associative rings, alternative division rings, or alternative radical rings. However, the Cayley-Dickson algebra, called a vector-matrix algebra by Zorn (Abh. Math. Sem. Hamburgischen Univ. vol. 8 (1930) p. 144), is not a special alternative ring. (Received July 12, 1949.)

496. Alfred Tarski: *A fixpoint theorem for lattices and its applications.* Preliminary report.

Let L be a complete lattice with inclusion relation $\leqq$, and $F(L)$ the family of all increasing functions on L to L. Theorem I. (i) *Given $f\in F(L)$, the set S of all $x\in L$ with $f(x)=x$ is nonempty and forms a complete lattice under $\leqq$.* (ii) *The conclusion still holds if, G being a commutative subfamily of $F(L)$ ($fg=gf$ for all f, $g\in G$), S is the set of all $x\in L$ with $f(x)=x$ for every $f\in G$.* Taking for L a closed interval of real numbers, I(i) is improved: Theorem II. *If f, g are real functions on $[a, b]$, f increasing, g con-*

tinuous, $f(a) \rightarrow g(a)$, *and* $f(b) \leqq g(b)$, *then the set* S *of all* $x \in [a, b]$ *with* $f(x) = g(x)$ *is nonempty, continuously ordered, and has a minimum and a maximum.* Taking for L a complete Boolean algebra, I(i) leads to: Theorem III. *Given* $a, b \in L$ *and* $f, g \in F(L)$, *there are* $c, d \in L$ *with* $f(a-c) = d$ *and* $g(b-d) = c$. Several theorems on equality of set power (for example, Cantor-Bernstein theorem, mean value theorem) reduce directly to III. I, III are applicable to topological functions (closure, derivative, and so forth); for example, I gives the familiar theorem: every closed set has a largest perfect subset. Theorems I–III were found in 1939; I(i), III improve earlier results of Knaster-Tarski (Annales de la Société Polonaise de Mathématique vol. 6, p. 133). (Received June 27, 1949.)

497. R. M. Thrall: *On projectivities in a finite-dimensional modular lattice.* Preliminary report.

It is well known that any non-distributive modular lattice L contains at least one sublattice isomorphic to the five-element lattice $\{r, s, t, u, w\}$ where $r = s \cap t = s \cap u = t \cap u$ and $w = s \cup t = s \cup u = t \cup u$. Any such sublattice is called a *projective root* in L; if, in particular, s covers r, the sublattice is called a *prime objective root* (p.p.r.). The relationship between the set of p.p.r.'s and the set of projective prime quotients in L is studied, and a canonical form for projectivities is obtained. A Galois correspondence is set up between sublattices L of the lattice N of all subspaces of an n-dimensional vector space V and subalgebras A of the algebra T of all linear transformations on V. It is shown that the closed elements L under this correspondence possess a property called *projective closure* which is defined in terms of linear transformations induced by projectivities in L. (Received July 23, 1949.)

498. Leonard Tornheim: *Lattice packings in the plane without crossing arcs.*

It is known that the non-overlapping lattice packings of an open set K, or its closure $\overline{K}$, are in one-to-one correspondence with the admissible lattices of $K-K$, the set of points which are the difference of two points in K. For sets J which are composed of a finite number of arcs and open sets (or their closures), conditions on the set $J-J$ are found so that a lattice arrangement of J be non-overlapping, have no arcs intersect open sets, and have no arcs cross. One of these conditions results from the fact that the set $c-d$, where c and d are two crossing arcs, contains a neighborhood of the origin. (Received July 20, 1949.)

499. L. R. Wilcox: *On a certain relational equation.*

Let S be a finite set and, for every binary relation X on $S \times S$, let X_t be the smallest transitive relation containing X. The following results are obtained concerning the equation $X_t = R$, where R is a given transitive relation. If R is a partial ordering, the solutions X are exactly those relations for which $R_0 \subset X \subset R$, R_0 being the *prime* subrelation of R (aR_0b in case aRb and no c exists with aRc, cRb). If R is the universal relation $S \times S$, then every symmetric solution X is of the form $Y + Y^*$, where Y is the prime subrelation of a suitable partial ordering (Y^* being the transpose of Y). All nonsymmetric solutions are obtained in the form $Y+Z$ with $Z \subset Y^*$. A characterization is given for the relations Y and Z which yield solutions. Finally, if R is an equivalence relation, the problem is easily reduced to the preceding case (even if S is infinite). (Received July 18, 1949.)

Analysis

500. Joshua Barlaz: *A generalization of the Toeplitz problem of summability.*

Let A and B be two linear sequence-to-sequence summability methods applied to a series $\sum a_n$. With certain conditions on A and B, the author proves that if $\sum a_n$ is A-summable, then the B transform of $\sum a_n$ is also A-summable. (In the Toeplitz problem, A is the identity transform.) The results of St. Mazur (Math. Zeit. vol. 28 (1928) p. 599), determining conditions for one summability method to be more powerful than another, are used. A systematic application of the results is made to Nörlund means and to some Riesz means. Special theorems where A is a sequence-to-function transform are also obtained. (Received July 14, 1949.)

501. E. F. Beckenbach: *A class of mean value functions.*

For a set of positive values $(a_1, a_2, \cdots, a_n)$, a mean value function is defined by $\mathfrak{N}_t(a) \equiv \sum a_j^t / \sum a_k^{t-1}$, and its properties are investigated. It is shown in particular that $\mathfrak{N}_t(a)$ increases continuously from min (a) to max (a) as t increases from $-\infty$ to $+\infty$, and that $\mathfrak{N}_t(a)$ satisfies the triangle inequality for $1 \leqq t \leqq 2$, and satisfies the same inequality with sign reversed for $0 \leqq t \leqq 1$. (Received July 14, 1949.)

502t. Lipman Bers: *The expansion theorem for sigma-monogenic functions.*

A Σ-monogenic function $f = u(x, y) + iv(x, y)$ is a complex-valued function $f = u(x, y) + iv(x, y)$ such that u and v satisfy the equations (1) $\sigma_1(x)u_x = \tau_1(y)v_y$, $\sigma_2(x)u_y = -\tau_2(y)v_y$ $(\sigma_1\sigma_2\tau_1\tau_2 > 0)$. Special Σ-monogenic functions, called "formal powers," can be obtained by quadratures (L. Bers and A. Gelbart., Trans. Amer. Math. Soc. vol. 56 (1944) pp. 67–93). The expansion theorem states that in the neighborhood of a point of regularity any Σ-monogenic function can be expanded in a "formal power series." This has been proved previously (loc. cit.) under the hypothesis that $\sigma_i(x)$, $\tau_j(y)$ are analytic functions. In the present paper the proof is given for the case where these functions are twice continuously differentiable. The expansion theorem leads to very precise information on the regularity properties of solutions of elliptic partial differential equations. Example: if $f(y) \in C'$, then any solution of $\Delta u + f(y)u_y = 0$ is analytic in x. (Received July 18, 1949.)

503. H. D. Block: *An integral transformation.* Preliminary report.

Consider the family $(n = 0, 1, 2, \cdots)$ of integral transformations on a suitable function $g(y)$, defined by $f_n(x, k) = \int_{-\infty}^{\infty} |x-y|^n e^{-k|x-y|} g(y)dy = L_n[g(y)]$. Since $f_n(x+a, k) = L_n[g(y+a)]$, $D_x f_n(x, k) = L_n[D_y g(y)]$ if $g(y) \in C, \in C^1$ in sections. Corresponding formulas are given if $g(y), g'(y)$ have finite jumps, and so on. For a fixed value of k we have for the successive application $L_m(L_n) = \sum_{i=0}^{n} C_{n,i}((m+i)!/(2k)^{m+i+1}) \cdot L_{n-i} + \sum_{j=0}^{m} C_{m,j}((n+j)!/(2k)^{n+j+1}) L_{m-j} + (m!n!/(m+n+1)!) L_{m+n+1} = L_n(L_m)$. For computation of the iterated kernels this formula may be simplified. Note also that L_n can be written as a linear combination of the iterates of L_0. This shows that all the eigen-functions of L_0 are eigen-functions of L_n (continuous spectrum). From previous results on the corresponding integral equations the converse is not true. These results also give the inversion of the transform (k fixed). Allowing k to vary one has the inversion formula $\text{Lim}_{k \to \infty} (k^{n+1}/n!) L_n[g(y)] = g(x+) + g(x-)$, thus restricting the class of possible transform functions. Further properties of the transformation are being studied. (Received July 18, 1949.)

504*t*. A. P. Calderón: *On a theorem of Marcinkiewicz and Zygmund.*

Let $F(x_1, x_2, \cdots, x_n)$ be a function harmonic for $x_n>0$ and $x_1, x_2, \cdots, x_{n-1}$ arbitrary. Suppose that every point of a measurable set E of $x_n=0$ is the vertex of a cone, otherwise contained in $x_n>0$, where the function is bounded for x_n small. Then, for almost every point Q of E, the integral of x_n^{-n+2} grad^2 F extended over any non-tangential neighborhood of Q is finite. This generalizes a result of Marcinkiewicz and Zygmund concerning harmonic functions of two variables. (Received August 2, 1949.)

505*t*. A. P. Calderón: *On the behaviour of harmonic functions at the boundary.*

(1) Let $F(x, y, z)$ be a harmonic function for $z>0$ and x, y arbitrary. Suppose that for every point $(x, y, 0)$ belonging to a set E of positive two-dimensional measure there is a cone with vertex at that point and otherwise contained in the half-space $z>0$, and such that within the cone and near to the point F is bounded. Then at almost every point of E the function F has a non-tangential limit. This is an extension to three dimensions of a well known result of Priwaloff. The proof holds for any number of dimensions, and is extensible to functions harmonic in groups of variables and in particular to regular functions of several complex variables. (2) Let $F(z_1, z_2)$, where $z_1=x_1+iy_1$, $z_2=x_2+iy_2$, be regular for $y_1>0$, $y_2>0$. Then at almost every point (x_1, x_2) of the plane $y_1=y_2=0$, either the function has a finite non-tangential limit, or the range of the values of F at every non-tangential neighborhood of (x_1, x_2) is dense in the whole complex plane. Again this result is valid for regular functions of n complex variables. For $n=1$ it reduces to a well known result of Plessner. (Received August 2, 1949.)

506*t*. M. M. Day: *Means and ergodicity of semigroups.*

A bounded semi-group $\mathfrak{S}$ of linear operators from a Banach space B to itself is called ergodic if there exists a directed system $\mathcal{A}$ of averages A such that $\lim_A (AS-A) = \lim_A (SA-A)=0$ for every S in $\mathfrak{S}$; we have three strengths of ergodicity of $\mathfrak{S}$ according as uniform, strong, or weak convergence is used in the operator algebra. It is shown that weak ergodicity of every bounded representation of Σ is equivalent to weak ergodicity of the right and left representations of Σ by right and left translations on $m(\Sigma)$, and equivalent to Σ amenable (see abstract 507). Uniform ergodicity is similarly related to strong amenability of Σ. (Received July 5, 1949.)

507. M. M. Day: *Means on semigroups and groups.*

A mean μ on a semigroup Σ is a positive linear functional of norm one on the space $m(\Sigma)$ of bounded, real-valued functions on Σ. Σ is called amenable if there exists a μ-invariant under all right- and left-translations by elements of Σ. Theorem 1. If Σ is the union of an expanding directed system of amenable subsemigroups, then Σ is amenable. Theorem 2. If G is a group and there is a left-invariant mean, then G is amenable. Theorem 3. If G is a group and H a normal subgroup of G, and if H and G/H are amenable, so is G. Results like Theorems 2 and 3 also hold if amenability is replaced by strong amenability: G is strongly amenable if there exists a directed system

Γ of finite means γ such that $\lim_{\gamma \in \Gamma} \|T\gamma - \gamma\|_{l_1} = 0$ for every right- or left-translation T of $l_1(G)$. (Received July 5, 1949.)

508. Aryeh Dvoretzky: *Bounds for the coefficient of univalent functions.*

Let (1) $f(z) = z + a_2 z^2 + \cdots + a_n z^n + \cdots$ be regular and univalent in $|z| < 1$ and denote by W the domain in the w-plane onto which $|z| < 1$ is mapped through $w = f(z)$. It is well known that, as $n \to \infty$, always $a_n = O(n)$ while $a_n = o(n^{-1/2})$ whenever W is bounded. This and similar facts suggest that limitations of a geometrical nature on the "extension" of W affect strongly the growth of the coefficients of $f(t)$. However this "extension" has little to do with the area of W. In the present paper it is gauged by means of the function $A(R) = \max_{|w|=R} \min_{w' \notin W} |w' - w|$, that is, the radius of the largest circle with center on $|w| = R$ the whole interior of which is contained in W. $A(R)$ is continuous for all $0 \leqq R < \infty$ and satisfies $0 \leqq A(R) \leqq R+1$. A general result giving bounds for a_n in terms of $A(R)$ is obtained. We quote here two very special applications: (1) There exists a finite positive contant K such that $\lim \sup_{n=\infty} |a_n|/n \leqq KL$ holds for all functions (1) satisfying $\lim \sup_{R=\infty} A(R)/R \leqq L$. (2) If W is such that it does not fully cover arbitrarily large circles, then $a_n = o(\log n)$ as $n \to \infty$. The methods extend to p-valent or mean p-valent functions and the results may be strengthened when special classes of univalent functions are considered. (Received July 15, 1959.)

509. W. F. Eberlein: *Banach-Hausdorff limits.* I.

A Banach-Hausdorff (B-H) functional is defined to be a linear functional $L(x)$ on the space m of bounded real sequences x which satisfies the four conditions for the ordinary Banach limit and, in addition, the fifth condition: $L(Hx) = L(x)$, where H is any regular Hausdorff transformation. The existence and structure of B-H functionals is determined, plus the domain of uniqueness—that is, the class of sequences x on which all B-H functionals coincide. (Received August 30, 1949.)

510. G. C. Evans: *An infinitely-valved harmonic function with branch curves of order two.* Preliminary report.

Given two closed curves in space, s_1, s_2, capped respectively by surface films D_1, D_2 of negligible surface tension, which are conductors, what positions will those surfaces assume if they are held respectively at potentials V_1 and V_2? The space is equivalent topologically to one in which the curves are distinct unlinked circles; the curves are of zero capacity. Let H be an infinitely leaved space of which the leaves H^j are pinned together in pairs alternately on s_1, s_2 as branch curves, H^{2k} with H^{2k+1} on s_1, and with H^{2k-1} on s_2, $-\infty < k < \infty$; and $v(M)$ be uniform and harmonic on H, bounded on $\{H^j\}$ for j bounded, with preassigned values at ∞, $v(M^{2k}) \to 2k(V_1 - V_2)$, $v(M^{2k+1}) \to 2V_1 + 2k(V_1 - V_2)$. Such functions exist with arbitrary values h_j on H^j at ∞ provided that the series $\sum |h_j| m^{|j|}$, from $-\infty$ to $+\infty$, converges for m a certain constant less than 1, and are uniquely determined in a class of functions which do not increase too rapidly with $|j|$. The D_1 and D_2 are respectively the loci $v(M^0) = v(M^1)$ and $v(M^0) = v(M^{-1})$ and are bounded unless V_1 or V_2 vanishes. The H degenerates into a finite number of leaves, in that case two, if and only if $V_1 = V_2$. (Received July 18, 1949.)

511*t*. L. Fejér and Gabor Szegö: *On special conformal mappings.*

1. Two proofs are given for the following theorem. Let $f(z) = u(r, \theta) + iv(r, \theta)$ be

regular for $r \leqq R$, real for real z, and $\partial u(r, \theta)/\partial\theta < 0$ for $r = R$, $0 < \theta < \pi$. Then for the same r and θ we have $v(r, \theta) > 0$. 2. Let $\sum_1^\infty b_n r^n \sin n\theta > 0$ for $r = 1$, $0 < \theta < \pi$. Then the same is true for $r < 1$, $0 < \theta < \pi$. This assertion is generalized by replacing r^n by λ_n where the sequence $\{\lambda_n\}$ is monotonic of order 4. 3. Let $S_n^{(k)}(z)$ be the Cesàro sum of order k and degree n of the geometric series $\sum_0^\infty z^n$. Writing $S_n^{(k)}(e^{i\theta}) = u_n(\theta) + iv_n(\theta)$, one has $(u_n(\theta) - u_n(0))v_n'(\theta) - u_n'(\theta)v_n(\theta) = (n+1)(1 - \cos\theta)(P_{n-1}^{(2)}(\cos(\theta/2)))^2/2$. Moreover writing $S_n^{(3)}(e^{i\theta}) = u_n(\theta) + iv_n(\theta)$ one has $u_n'(\theta)v_n''(\theta) - u_n'(\theta)v_n'(\theta) = (n+2)(P_{n-1}^{(3)}(\cos(\theta/2)))^2/3$. Here $P_{n-1}^{(\lambda)}$ denotes the ultraspherical polynomials, that is, the coefficient of w^{n-1} in the expansion of $(1 - 2\cos(\theta/2)\cdot w + w^2)^{-\lambda}$. These results are refinements of certain results of Egerváry (Math. Zeit. vol. 42 (1937)). 4. A power series $f(z)$ is constructed with totally monotonic coefficients such that the mapping $w = f(z)$ of the unit circle is not convex. (Received July 27, 1949.)

512. D. B. Goodner: *Projections in normed linear spaces.*

A normed linear space X has property P_s, $s \geqq 1$, if and only if for every normed linear space Y containing X there is a projection T, $\|T\| \leqq s$, of Y onto X. *Theorem* 1. If a normed linear space X has property P_1 and an extreme point on its unit sphere C, then X is equivalent to the space of real-valued continuous functions on some extremally disconnected compact Hausdorff space H. *Theorem* 2. If an abstract (M)-space has property P_1, then it has a unit element. (Received July 14, 1949.)

513*t*. L. M. Graves: *On the definition of generalized curves.*

Let $I = [0 \leqq t \leqq 1]$, E be a Euclidean space, and S the unit sphere in E. Let $H =$ [all $F(y, r)$ continuous for $y \in E$, $r \in S$]. A representation ρ consists of a point $y(0)$ in E and a real function $v(t, F)$ on $I \times H$, such that: 1) $v(t, F)$ is A.C. on I; 2) $v(t, F)$ is linear on H; 3) if $F \geqq 0$, $v(t, F)$ is nondecreasing in t; 4) $v(0, F) = 0$; 5) if $F_i(r) \equiv r^i$, $y^i(t) \equiv y^i(0) + v(t, F_i)$, and $F(y(t), r) = 0$ for $r \in S$, $t_1 \leqq t \leqq t_2$, then $v(t, F)$ is constant on $t_1 \leqq t \leqq t_2$. Let π consist of all $\theta(t)$ which are increasing, quasilinear, and map I into itself. Let H_1 consist of all F in H which satisfy a Lipschitz condition with constant one and $|F(y, r)| \leqq 1$. For two representations $\rho_1 = (y_1(0), v_1)$, $\rho_2 = (y_2(0), v_2)$, let $\beta(v_1, v_2, \theta) =$ l.u.b. $|v_1(t, F) - v_2(\theta(t), F)|$ for $t \in I$, $F \in H_1$, and let the distance $\delta(\rho_1, \rho_2) = \|y_1(0) - y_2(0)\| +$ g.l.b. $\beta(v_1, v_2, \theta)$ for all $\theta \in \pi$. A generalized curve is a class of representations whose mutual distances are zero. In the space of generalized curves, the desired compactness theorem now follows at once from the theorem of Ascoli for mappings between compact metric spaces. There is a correspondence between the representations introduced here and those used by McShane (Duke Math. J. vol. 6 (1940) pp. 513–536). The approach outlined here gives a simpler aspect to the basic part of the theory. (Received July 18, 1949.)

514. William Gustin: *On decomposition of an interval into finitely many congruent sets.*

Let G be an additive subgroup of the real numbers; and let I be an interval of G. It is shown that if I can be decomposed into a finite number (>1) of congruent disjoint subsets, then I can also be decomposed into a finite number (>1) of congruent disjoint intervals. As a consequence no open or closed interval of real numbers can be decomposed into finitely many disjoint congruent sets. (Received July 14, 1949.)

515. Fritz Herzog and George Piranian: *Sets of convergence of Taylor series.*

Let $\sum_n a_n z^n$ be a Taylor series of radius of convergence one. The totality of those points on the unit circle C at which the Taylor series converges will be called its *set of convergence*. It was shown by Neder that every closed arc on C is the set of convergence of some Taylor series. Mazurkiewicz extended this result by showing that every closed set on C, as well as every open set on C, is the set of convergence of some Taylor series. In the present paper the authors obtain the result that every set of type F_σ on C is the set of convergence of some Taylor series. It is known, on the other hand, that every set of convergence is necessarily of type $F_{\sigma\delta}$. Another result obtained in the paper states that if M is any closed set on C, there exists a Taylor series which converges *uniformly* on M and diverges on $C-M$. This is the best result in the sense that if a Taylor series converges uniformly on its set of convergence then this set of convergence must necessarily be closed. (Received February 23, 1949.)

516*t*. Edwin Hewitt and H. S. Zuckerman: *Certain rings related to locally compact groups*. Preliminary report.

Let G be a locally compact topological group. Let $\mathfrak{C}_{\infty\infty}(G)$ be the linear space of all complex continuous functions on G each of which vanishes outside of some compact set. Let $x \rightarrow U_x$ be an irreducible weakly continuous representation of G by unitary operators U_x on a Hilbert space $\mathfrak{H}$ having an orthonormal basis $\{\phi_\gamma\}$, $\gamma \in \Gamma$. Let $\mathfrak{A}(G)$ be the set of all linear combinations of functions $(U_x\phi_\gamma, \phi_\delta)$, for all possible representations $x \rightarrow U_x$ of the type described. Let $\mathfrak{R}_u(G)$ be the set of all complex linear functionals on $\mathfrak{C}_{\infty\infty}(G)$ which are continuous in the uniform topology for $\mathfrak{C}_{\infty\infty}(G)$; let $\mathfrak{S}_u(G)$ be the set of all complex linear functionals on $\mathfrak{A}(G)$ continuous in the uniform topology for $\mathfrak{A}(G)$; and let $\mathfrak{S}_a(G)$ be the set of all complex linear functionals on $\mathfrak{A}(G)$ continuous in the norm $|||\sum_{j=1}^n \alpha_j f_j||| = \sum_{j=1}^n |\alpha_j|$. Each of these sets of functionals becomes a ring under the usual definition of sum and under the definition $MN(f) = \iint_{G\times G} f(xy)d\mu(x)d\nu(y)$, where μ and ν are the measures corresponding to M and N respectively. All three rings are extensions of the L_1-algebra of G, containing this algebra as a closed two-sided ideal. All three rings contain isomorphs of G, are commutative if and only if G is Abelian, and have identity elements. If all the representations $x \rightarrow U_x$ described above are finite-dimensional, then $\mathfrak{R}_u(G)$ and $\mathfrak{S}_u(G)$ have the property that the intersection of all maximal two-sided ideals is the nil-ideal. $\mathfrak{R}_u(G)$, $\mathfrak{S}_u(G)$, and $\mathfrak{S}_a(G)$ all have nil Jacobson radical. The complete structure of maximal ideals is described for certain special cases. (Received September 8, 1949.)

517. Edwin Hewitt and H. S. Zuckerman: *Certain rings related to locally compact rings*. Preliminary report.

Let G be a locally compact topological group. Let $\mathfrak{C}_{\infty\infty}(G)$ be the linear space of all complex continuous functions on G each of which vanishes outside of some compact set. Let $x \rightarrow U_x$ be an irreducible weakly continuous representation of G by unitary operators U_x on a Hilbert space $\mathfrak{H}$ having an orthonormal basis $\{\phi_\gamma\}_{\gamma\in\Gamma}$. Let $\mathfrak{A}(G)$ be the set of all linear combinations of functions $(U_x\phi_{\gamma_1}, \phi_{\gamma_2})$, for all possible representations $x \rightarrow U_x$ of the kind described. Let $\mathfrak{R}(G)$ be the set of all complex linear functionals on $\mathfrak{C}_{\infty\infty}(G)$ which are linear combinations of non-negative functionals; let $\mathfrak{R}_u(G)$ be the set of all functions in $\mathfrak{R}(G)$ continuous in the uniform topology for $\mathfrak{C}_{\infty\infty}(G)$; let $\mathfrak{S}_u(G)$ be the set of all complex linear functionals on $\mathfrak{A}(G)$ which are continuous in the uniform topology for $\mathfrak{A}(G)$; and let $\mathfrak{S}_a(G)$ be the set of all complex linear functionals on $\mathfrak{A}(G)$ continuous in the norm $|||\sum_{j=1}^n \alpha_j f_j||| = \sum_{j=1}^n |\alpha_j|$. Each of these sets of functionals becomes a ring under the definitions $MN(f) = \iint_{G\times G} f(xy)d\mu(x)d\nu(y)$

where μ and ν are the measures corresponding to M and N respectively, and $(M+N)(f) = M(f)+N(f)$. If G is compact, then $\mathfrak{R}(G)=\mathfrak{R}_u(G)=\mathfrak{S}_u(G)$, while $\mathfrak{S}_a(G)$ is in general a super-ring of the others. All four rings are extensions of the L_1-algebra $\mathfrak{L}_1(G)$; in the uniform topologies for $\mathfrak{R}_u(G)$, $\mathfrak{S}_u(G)$, and $\mathfrak{S}_a(G)$, $\mathfrak{L}_1(G)$ is closed. All four of the rings under discussion contain isomorphs of G, are commutative if and only if G is Abelian, and have identity elements. $\mathfrak{R}_u(G)$, $\mathfrak{S}_u(G)$, and $\mathfrak{S}_a(G)$ have zero Jacobson radical. The complete structure of maximal ideals is described for particular cases. (Received July 8, 1949.)

518*t*. Einar Hille: *On the differentiability of semi-group operators.*

Let $T(\xi)$ be a semi-group operator such that (i) $T(\xi_1+\xi_2)=T(\xi_1)T(\xi_2)$, (ii) $\|T(\xi)\| \leqq 1$, (iii) $\lim_{\xi\to 0}(\xi)x=x$ for all x in the operand space and let A be its infinitesimal generator with resolvent $R(\lambda; A)$ existing at least for $\mathfrak{R}(\lambda)<0$. $T'(\xi)=AT(\xi)$ is ordinarily unbounded; this note contains a study of what conditions must hold in order that $T'(\xi)$ shall exist as a bounded operator for each $\xi<0$. If $T'(\xi_0)$ is bounded, so is $T^{(n)}(\xi)$ for $\xi \geqq n\xi_0$, $n=1, 2, 3, \cdots$. Lower limits may be found for the rate of growth of $g(\xi)=\|T'(\xi)\|$ as $\xi\to 0$ but no upper ones. Knowing $g(\xi)$ we can estimate the distance $\delta(\tau)$ of the spectrum of A from the points of the line $\mathfrak{R}(\lambda)=1$. In particular, if $\log g(\xi)\in L(0, 1)$, then $R(\tau)=\|R(1+i\tau; A)\|\,(1+|\tau|)^{-1}\in L(-\infty, \infty)$. Conversely, if $R(\tau)\in L(-\infty, \infty)$ and $\delta(\tau)$ is fairly regular, then $T(\xi)$ has bounded derivatives of all orders for $\xi<0$. (Received August 16, 1949.)

519*t*. Einar Hille: *On the integration problem for Fokker-Planck's equation in the theory of stochastic processes.*

A temporally homogeneous one-parameter stochastic process leads to the partial differential equation $[b(x)U]_{xx}-[a(x)U]_x=U_t$ where $b(x)>0$. The integration problem consists in finding for what functions $a(x)$ and $b(x)$ the equation has a unique solution $U(x, t)$, $t>0$, in $L(-\infty, \infty)$ which converges in the mean of order one to a preassigned function $g(x)$ in $L(-\infty, \infty)$ when $t\to 0$. K. Yosida has recently shown that this problem is equivalent to finding when the differential operator $A[f]=[b(x)f(x)]''-[a(x)f(x)]'$ generates a semi-group of translation operators in $L(-\infty, \infty)$. The present paper gives a simplified discussion of the problem and leads to less restrictive conditions than those of Yosida. $A[f]$ generates contraction operators if merely $b''(x)-a'(x)$ is bounded above. If in addition, for instance, $|b'(x)-a(x)| \leqq K(|x|+1)$ and the integral of $t/b(t)$ diverges over the ranges $(-\infty, 0)$ and $(0, \infty)$, then the equation defines a stochastic process. (Received August 16, 1949.)

520*t*. Shizuo Kakutani: *Concrete representation of Brownian motions.*

Let $(\Omega, \mathfrak{B}, \mu)$ be a probability space (that is, a measure space with $\mu(\Omega)=1$), and $(S, \mathfrak{M}, m)$ a measure space with $m(S)\leqq\infty$. Let $\mathfrak{M}_0=\{M \mid M\in\mathfrak{M}, 0<m(M)<\infty\}$. A real-valued function $x(M, \omega)$ $(M\in\mathfrak{M}_0, \omega\in\Omega)$ is a generalized Brownian motion if (i) for any fixed $M\in\mathfrak{M}_0$, $x_M(\omega)=x(M, \omega)$ is a Gauss function (that is, measurable in ω and has a Gaussian distribution with mean 0) with variance $m(M)$, (ii) if M_k, $k=1, \cdots, n$, are disjoint, then $x(M_k, \omega)$, $k=1, \cdots, n$, are independent and $x(\bigcup_{k=1}^n M_k, \omega)=\sum_{k=1}^n x(M_k, \omega)$ a.e. on Ω. If $(S, \mathfrak{M}, m)$ is the ordinary Lebesgue measure space on the infinite real line $S=\{s \mid -\infty<s<\infty\}$, and if we consider only finite intervals $I=\{s \mid a<s\leqq b\}$ as $M\in\mathfrak{M}_0$, then $x(M, \omega)$ is a Brownian motion in the sense of N. Weiner. It is shown that every generalized Brownian motion determines a linear

and isometric embedding $f(s)\to x_f(\omega)$ of $L^2(S)$ into $L^2(\Omega)$ as a Gauss subspace (that is, a closed linear subspace of $L^2(\Omega)$ consisting only of Gauss functions) such that $f_M(s)\to x(M, \omega)$, where $f_M(s)$ is the characteristic function of M. Conversely, every such embedding of $L^2(S)$ into $L^2(\Omega)$ determines a generalized Brownian motion. Properties of Brownian motions are discussed by using this concrete representation. (Received July 14, 1949.)

521t. Shizuo Kakutani: *Determination of the spectral type of the flow of Brownian motion.*

Let $x(M, \omega)$ be a generalized Brownian motion as defined in the preceding abstract. Assume that the ring generated by $\{x_M(\omega) \mid M\in\mathfrak{M}_0\}$ is dense in $L^2(\Omega)$. Then, for any flow (=one-parameter group of measure preserving transformations) $\{\psi_t(s)\}$ on $(S, \mathfrak{M}, m)$, there corresponds a flow $\{\phi_t(\omega)\}$ on $(\Omega, \mathfrak{B}, \mu)$ such that $x(\psi_t(M), \omega) = x(M, \phi_t(\omega))$ a.e. on Ω for any $M\in\mathfrak{M}_0$ and for any t. If $(S, \mathfrak{M}, m)$ is the ordinary Lebesgue measure space on the infinite real line $S=\{s \mid -\infty<s<\infty\}$ and if $\psi_t(s)=s+t$, then the corresponding flow $\{\phi_t\}$ is the flow of Brownian motion discussed by N. Wiener. The main purpose of this paper is to determine the spectral type of the one-parameter group of unitary transformations $\{U_t\}$ defined on $L^2(\Omega)$ by $U_tx(\omega) = x(\phi_t(\omega))$. It is shown that $L^2(\Omega)$ is decomposed into a direct sum of a countable number of closed linear subspaces $\mathfrak{M}_n$, called the homogeneous chaos of order n, each $\mathfrak{M}_n$ invariant under U_t, in such a way that $\{U_t\}$ on $\mathfrak{M}_n$ is spectrally isomorphic with $\{V_t^{(n)}\}$ on $L_s^2(S^{(n)})$, where $L_s^2(S^{(n)})$ is the L^2-space of all real-valued measurable functions $f(s_1, \cdots, s_n)$ defined on $S^{(n)}=S\times\cdots\times S$ (n times) and symmetric in its n variables, such that $\int\int_{S\times\cdots\times S}|f(s_1, \cdots, s_n)|^2 ds_1\cdots ds_n<\infty$, and $\{V_t^{(n)}\}$ on $L_s^2(S^{(n)})$ is defined by $V_t^{(n)}f(s_1, \cdots, s_n)=f(\psi_t(s_1), \cdots, \psi_t(s_n))$. This determines completely the spectrum of $\{U_t\}$. The construction of the homogeneous chaos $\mathfrak{M}_n$ is obtained by using the Hermite polynomials $H_n(u, \sigma)=(n!)^{-1}(-1)^n(\sigma^n e^{u^2/2\sigma})d^n(e^{-u^2/2\sigma})/du^n$ and an essential use is made of the following functional relation: $H_n(u+v, \sigma+\tau) = \sum_{k=0}^{n} H_k(u, \sigma)H_{n-k}(v, \tau)$. (Received July 14, 1949.)

522. R. E. Lane: *Foundations of the theory of continued fractions.* I.

The author lays the foundations for a general theory of the continued fraction with bounded sequence $\{f_n\}_0^\infty$ of approximants by establishing the existence of a nest R_{n+1} of finite circular regions such that (i) for each positive integer, n, R_n is the smallest circular region containing f_{n-1} and R_{n+1} and (ii) the region R common to all of the R_n is the smallest circular region which contains all sequential limit points of $\{f_n\}_0^\infty$. This fundamental result implies, for example, that every convergent continued fraction is positive definite or can be transformed, by a contraction and an equivalence transformation, into a positive definite continued fraction. The theorem provides a sequence of inequalities, each involving a partial numerator and two partial denominators of the continued fraction, necessary and sufficient for the sequence $\{f_n\}_0^\infty$ to be bounded. In addition, it provides a relatively simple algebraic condition which is necessary and sufficient for convergence. (Received July 18, 1949.)

523. C. E. Langenhop and A. B. Farnell: *The existence of forced periodic solutions of second order differential equations near certain equilibrium points of the unforced equation.*

The equation $\ddot{x}+a\dot{x}+x+x^2=k\cos\omega t$, $a>0$, $k>0$, is discussed in detail. This equation is replaced by a system ($\dot{x}=y$, $\dot{y}=f(x, y, t)$) the solutions of which define a mapping T of the (x, y) plane into itself. This mapping is obtained by replacing t by $t+2\pi/\omega$ in every solution of the system. For k sufficiently small a closed convex region Ω can be constructed from the solutions of certain associated linear systems so that Ω is mapped into itself under T. Brouwer's fixed point theorem is then utilized to assert the existence of at least one periodic solution of the system in Ω. Another region Φ surrounding Ω is obtained and its boundary is shown to have the index zero with respect to the vector field $[P_0P_1]^{\rightarrow}(P_1=TP_0)$, which implies the existence of another periodic solution in $\Phi-\Omega$. The method of obtaining Ω above is then extended to prove that for k sufficiently small there are periodic solutions of the equation $\ddot{x}+f(x)\dot{x}+g(x)=ke(t)$ in the neighborhood of points $(x={}_0x, \dot{x}=0)$ such that $g(x_0)=0$, $g'(x_0)>0$, $f(x_0)\neq 0$, if $f(x)$, $g(x)$, $e(t)$ are analytic and $e(t)$ is periodic. (Received June 27, 1949.)

524*t*. Joseph Lehner: *Note on power series with integral coefficients.*

H. Graetzer has proved (J. London Math. Soc. (1947) pp. 90–92) that any complex number x_0 with $|x_0|<1$ is a root of a power series with rational integral coefficients and radius of convergence unity. The author proves the following generalization: Let E be a definite or countable set of complex numbers all lying in the interior of the unit circle; $\bar{E}$ is the set of complex conjugates. E shall have no limit points interior to the unit circle. Then there is a power series, with rational integral coefficients and radius of convergence unity, which vanishes on E and $\bar{E}$, and nowhere else inside the unit circle. (Received July 11, 1949.)

525*t*. Walter Leighton: *On the detection of the oscillation of solutions of a second order linear differential equation.*

The principal result of this paper is the following: Let (1) $(r(x)y')'+p(x)y=0$ be a differential equation in which r, r', and p are continuous and $r>0$ on $0<x<\infty$. If $p(x)>0$ near $x=+\infty$ and if $\int_1^\infty r^{-1}dx=\int_1^\infty p dx=+\infty$, every solution of (1) has an infinity of zeros on the interval $1<x<\infty$. Similarly, if $p>0$ near $x=0+$ and if $\int_0^1 r^{-1}\,dx=\int_0^1 p\,dx=+\infty$, every solution of (1) has an infinity of zeros on the interval $0<x<1$. The paper will be published in the Duke Mathematical Journal. (Received July 15, 1949.)

526. C. B. Morrey: *Quasi-convexity and the lower semicontinuity of double integrals.*

Assume $f(u, v, \mathbf{x}, p, q)(\mathbf{x}=x, \cdots, x^N)$ continuous, D a domain, $I(\mathbf{x}, D)$ the double integral of f. $\phi(p, q)$ is called *quasi-convex* if linear functions minimize its integral. The following are proved: (1) A necessary and sufficient condition that $I(\mathbf{x}, D)\leqq\lim\inf I(\mathbf{x}_n, D)$ whenever $\mathbf{x}_n \rightrightarrows \mathbf{x}$ with $|\mathbf{x}_{nu}|$ and $|\mathbf{x}_{nv}|$ uniformly bounded is that f be quasi-convex in (p, q). (2) A necessary condition that f be quasi-convex in (p, q) is that $f(u_0, v_0, \mathbf{x}_0, p_0+\lambda\boldsymbol{\xi}, q+\mu\boldsymbol{\xi})$ be convex in $\boldsymbol{\xi}$ and in (λ, μ); if f is of class C', this is the Weierstrass condition. (3) A sufficient condition that $\phi(p, q)$ be quasi-convex in (p, q) is that, for each (p_0, q_0), there are constants A_i, B_i, and $C_{ij}(C_{ji}=-C_{ij})$ such that $\phi(p_0+\boldsymbol{\xi}, q_0+\boldsymbol{\eta})\geqq\phi(p_0, q_0)+A_\alpha\xi^\alpha+B_\alpha\eta^\alpha+C_{\alpha\beta}\xi^\alpha\eta^\beta$. (4) If f is quasi-convex, of class C', and homogeneous of degree 2 in (p, q) with $m(p^2+q^2)\leqq f\leqq M(p^2+q^2)(m>0)$ then $I(x, D)$ is lower-semi-continuous with respect to weak convergence in $\mathfrak{P}_2$. (5) With the hypotheses in (4), there is a continuous minimizing vector with given boundary values. (1) and (2) generalize to k-tuple integrals. (Received July 18, 1949.)

527*t*. Helen K. Nickerson: *On overconvergence of Dirichlet series.*

Let $f(s)$, where $s=\sigma+it$, denote the function defined by the Dirichlet series $\sum_{n=1}^{\infty} a_n \exp(-\lambda_n s)$ (assumed to have finite abscissa of ordinary convergence) and its analytic extension. For each integer $r>1$ an absolutely convergent integral formula is derived, representing the partial sums (remainders) of the series in the half-plane $\sigma \leqq \eta_k (\sigma > \eta_k)$, where $f(s)$ is regular and $O(|t|^k)$ in $\sigma > \eta_k$ and $r>k$. Abscissae η_k surely exist for $k \geqq 1$. These formulae are used to prove the following theorem: If $l=\lim \sup_{p\to\infty}(1/\lambda_{n_p}) \log [1/(\lambda_{n_p+1}-\lambda_{n_p})[<+\infty$, then the partial sums corresponding to the indices $\{n_p\}$ converge to $f(s)$ in the half-plane $\sigma > \eta_k + kl$, for each value of k, uniformly on any closed bounded subset. Thus, if $l=0$ for a suitable subsequence of indices, $f(s)$ can be represented by a sequence of partial sums in the entire half-plane in which it is regular and of finite order; overconvergence due to Ostrowski gaps in the sequence of exponents $\{\lambda_n\}$ extends to this half-plane (and not merely to a suitable neighborhood of each point of the line of convergence in which the limit function is regular). (Received July 18, 1949.)

528*t*. Helen K. Nickerson: *On overconvergence of general series.*

The method of harmonic majorants, used by Bourion in the study of overconvergence for Taylor's series and by Walsh for general sequences with the same order of convergence as geometric series, is extended to more general series for which the convergence is weaker, as indicated by Bourion for Dirichlet series. In the general case the following results hold: overconvergence, when it occurs, is not local but extends to all points of the boundary of the region of convergence in which the limit function is regular (with exceptions possible in the case that the domain of convergence consists of two or more regions); the Ostrowski "gap" hypothesis, or a suitable analogue, implies overconvergence at all regular points of the boundary of the region of convergence (analogue of Hadamard gap theorem as corollary); non-gap overconvergence must, if the subsequence is not too sparse, occur at all points of the boundary of the region of convergence. The hypotheses used can be verified not only for Dirichlet series and factorial series, but also for Hermitian series, series of Laguerre polynomials, generalized factorial series and Newton series, a class of series studied by Carmichael, and the "generalized" Dirichlet series studied by Greenwood. (Received July 18, 1949.)

529*t*. Ruth E. O'Donnell: *A note on the location of the zeros of polynomials.*

Let $p(z)=\sum_{k=0}^{n} a_k z^k$ and form the ratios $-a_k/a_{k+1}$ $(k=0, \cdots, n-1)$. If no coefficient of $p(z)$ vanishes, then every half-plane containing the origin and at least one of the ratios also contains at least one zero of $p(z)$. If one or more of the coefficients vanish, then there is at least one zero of $p(z)$ in every half-plane containing the origin. If all the ratios lie in $\theta-\pi/h \leqq \arg z \leqq \theta+\pi/h$ $(h \geqq n)$, then all the zeros of $p(z)$ lie in $\theta-\pi[(n-1)/n+1/h] \leqq \arg z \leqq \theta+\pi[(n-1)/n+1/h]$ and all the k-fold zeros of $p(z)$ lie in $\theta-\pi[(n-k)/(n+k+1)+1/h] \leqq \arg z \leqq \theta+\pi[(n-k)/(n-k+1)+1/h]$. (Received August 7, 1949.)

530. A. M. Ostrowski: *On Theodorsen and Garrick's method of conformal mapping.*

A discussion of convergence features of the method for conformal mapping developed by Theodorsen and Garrick is given. The estimates given by Warschawski in the case of a specific initial approximation are further improved and developed under general assumptions about the initial approximation and the given contour. The corresponding convergence and existence theorems are given for a more general integral equation. (Received June 29, 1949.)

531. George Piranian and Barbara Piranian: *Analytic theory of Nörlund transformations.* Preliminary report.

The convergence field of the Nörlund transformation (N, p) is studied in terms of the function represented by the Taylor series (1) $\sum p_n z_n$. For the cases where (1) converges absolutely on the unit circle, a more or less explicit description of the convergence field is obtained. In particular, this description supplies a negative answer to the following question by G. M. Wing: If (N, p) is a regular Nörlund transformation stronger than convergence, does there necessarily exist another regular Nörlund transformation which is stronger than convergence but weaker than (N, p)? (Received July 16, 1949.)

532*t*. Tibor Rado and P. V. Reichelderfer: *On generalized Jacobians.*

Let T be a continuous mapping from a simply connected Jordan region R into Euclidean three-space E_3 which represents a surface having finite Lebesgue area. If F is a Cartesian frame of reference in E_3, then T gives rise to a series of rectangle functions, each of which possesses a derivative almost everywhere in the interior R_0 of R [T. Radó, *Length and area*, Amer. Math. Soc. Colloquium Publications, vol. 30, New York, 1948]. It is known that these derivatives satisfy certain relations at all points in R^0 except those belonging to a set $e(T, F)$ of measure zero and depending upon both T and F. The main purpose of this paper is to show that in a number of important cases, the set $e(T, F)$ may be replaced by a set $e(T)$ of measure zero and independent of F. These results are used to strengthen various theorems in the theory of surface area. (Received May 5, 1949.)

533*t*. L. L. Rauch: *Oscillation of a third order nonlinear autonomous system.*

In-the-large properties of the solutions of the system $\dot{x}=K_1[f(x)-(1+K_4)x-z]$, $\dot{y}=-K_2[f(x)-x-z]$, $\dot{z}=-K_3(y+z)$ are considered. K_1, K_2, K_3, and K_4 are independent positive constants and the nonlinearity is introduced by the function $f(x)$ which is assumed to be continuous and bounded above and below. In addition it is assumed that $xf(x)\geqq 0$ and that $f'(0)$ exists. When written as a single third-order equation in x the system becomes $k_1\dddot{x}+[k_2+k_3g(x)]\ddot{x}+k_3g'(x)\dot{x}^2+g(x)\dot{x}+x=0$ which is a generalization to the third order of Liénard's equation $\ddot{x}+g(x)\dot{x}+x=0$. By an application of Brouwer's fixed point theorem in the phase space (coordinates x, y, and z) it is proved that a sufficient condition for the existence of at least one period solution is $f'(0)>1+K_4+(1/2K_1)[K_2+K_3-((K_2-K_3)^2+4K_2K_4)^{1/2}]$ and $K_4>5.0K_1+9.7K_2+2.4K_1K_4+4.6\sup_{-\infty\leqq x\leqq\infty}f(x)/x$. It is also shown under this condition that after a long enough time every solution, with the exception of two singular cases, becomes a nondecaying oscillation which is not too far from the periodic solution or solutions in the phase space. It is established that the existence proof still holds when the system is generalized to $k_1\dddot{x}+[k_2+k_3g(x)]\ddot{x}+k_3g'(x)\dot{x}^2+g(x)\dot{x}+h(x)=0$. (Received July 14, 1949.)

534t. Francis Regan and Charles Rust: *On natural boundaries of a generalized Lambert series.*

In this paper the authors are interested in determining conditions of the coefficients of the series $F(z)=\sum_{n=1}^{\infty}(a_n b_n z^n)/(1-a_n z^n)$ introduced in 1932 by Feld, so that the function represented by the series will have a natural boundary. Special cases of this series have natural boundaries. If $a_n=a^n$, and if $b_n\equiv 1$ or if $0<b_n\leqq B<\infty$, the series represents a function which has the circle $|z|=1/|a|$ as a natural boundary. If $a_n=a$ and $|a|\leqq 1$ and if $b_n\equiv 1$, the series has the unit circle as a natural boundary. If a_n is such that $|a_n|\leqq 1$ and $\text{LUB}[|a_n|^{1/n}]=1$, this series and the Lambert series $L(z)=\sum_1^{\infty}(c_n z^n)/(1-z^n)$ represent the same function within the unit circle, provided $|c_n|\leqq C<\infty$, where $a_n b_n=\sum_{d|n}S(n/d)\sum_{d_1|d}c_{d_1}$ [$S(d)$ is a generalized Moebius function; Doyle, Ann. of Math. vol. 40, p. 357]. With these conditions a theorem for the Feld series analogous to Knopp's extension of the Franel theorem for the Lambert series is established; and we can show that if the sequences a_n and b_n are real and $0<c_n\leqq C<\infty$, the unit circle is a natural boundary for the function represented by the Feld series. (Received July 15, 1949.)

535. W. T. Reid: *Self-adjoint boundary value problems.*

This paper presents an analysis of self-adjoint boundary value problems consisting of a system of first-order linear ordinary differential equations and two-point boundary conditions, and which are definite, according to either the definition of Bliss [Trans. Amer. Math. Soc. vol. 44 (1938) pp. 413–428] or that of the author [Trans. Amer. Math. Soc. vol. 52 (1942) pp. 381–425]. In particular, there is given a canonical form for such boundary problems. (Received July 18, 1949.)

536. L. G. Riggs and W. T. Scott: *Partially bounded J-fractions.*

Let Y_p be a constant such that $F_p(x)=\sum_1^p\beta_n x_n^2-2\sum_1^{p-1}\alpha_n x_n x_{n+1}\geqq -Y_p\sum_1^p x_n^2$, for all values of x_n, where α_n, β_n, x_n are real. The J-forms $F_p(x)$ are called partially bounded if lim g.l.b. $Y_n=Y$ exists. A sufficient condition for partial boundedness of the $F_p(x)$ is $|\alpha_p|\leqq A$, $\beta_p\geqq B$. Application to the J-fraction $-\text{K}_1^{\infty}(-a_{p-1}^2/(z+b_p))$ of the method of Wall (Bull. Amer. Math. Soc. vol. 52 (1946) pp. 686–693) establishes the existence of a uniquely determined convex region corresponding to the J-fraction. The J-fraction converges uniformly in any closed region at a positive distance from the convex set. (Received July 18, 1949.)

537. E. K. Ritter: *Second differentials of functions in exterior ballistics.*

The equations of motion of the mass-center of a projectile ordinarily are integrated under simplifying "standard" assumptions, yielding "normal" trajectories as solutions. Calculation of actual effects of "disturbances" (departures from standard conditions) is tedious and costly; linear approximations thereto (called "differential effects" and based on a theory developed by G. A. Bliss during World War I) are more easily computed. In an attempt to meet a need for improved estimates, the present paper develops a mathematical theory of *second* differential effects of disturbances. It includes: (1) a proof of the existence of a second differential of the mapping defined by the equations of motion; (2) a demonstration that this second differential is expressible in terms of the second variations of the mass-center coordinates; and (3) a derivation of the differential system satisfied by these second variations. (Received July 22, 1949.)

538*t*. W. W. Rogosinski and Gabor Szegö: *Extremum problems on non-negative trigonometric polynomials.*

Let $\sum_1^n b_k \sin k\theta$ be a sine-polynomial of degree n which is not identically zero and non-negative in $0 \leqq \theta \leqq \pi$. Let $b_1 = 1$. Exact bounds are obtained for the coefficients b_n, b_{n-1}, b_2, b_3. General methods based on the theory of orthogonal polynomials are discussed in order to obtain such bounds in the case of any b_k, and these methods are illustrated by the cases b_4 and b_5. (Received July 27, 1949.)

539*t*. Herman Rubin: *An axiomatic approach to integration.*

Let $\mathcal{E}$ be a linear space of real-valued functions on a set X, E a positive linear functional (called elementary integral) on $\mathcal{E}$. Two cases are considered, according to whether for a sequence of positive functions whose sum is greater than some function, the sum of their integrals is necessarily greater than the integral of that function, or not. $\sum$ denotes a countable sum if the first case (countably additive) is assumed, a finite sum (finitely additive) otherwise. An $f \in \mathcal{E}$ is called an ϵ-approximant of g if there exist positive h_l such that $|f-g| \leqq \sum h_l$ and $\sum E(h_l) < \epsilon$. A function g is termed integrable if for every positive ϵ, g has an ϵ-approximant. Let $\mathcal{E}^*$ be the space of integrable functions. Then $\mathcal{E}^*$ is a lattice if for every $f \in \mathcal{E}$, $|f|$ has a positive ϵ-approximant for all ϵ. If $(X, \mathcal{E}, E)$ and $(Y, \mathcal{J}, F)$ are two such systems, then $(X \times Y, \mathcal{E} \times \mathcal{J}, E \times F)$ is also, and countable additivity and the lattice property are preserved. This leads to an elementary proof of Fubini's Theorem in both cases. (Received July 16, 1949.)

540*t*. Herman Rubin: *Measures and axiomatically defined integrals.*

Let $(X, \mathcal{E}, E)$ be a set, a linear space of functions on X, and an elementary integral on $\mathcal{E}$. Let $\mathcal{E}^*$ be a lattice. Then a function $\mathcal{J}$ is measurable if for any positive g in $\mathcal{E}^*$, mid $(-g, \mathcal{J}, g)$ is in $\mathcal{E}^*$. If 1 is measurable, a measure μ can be defined such that the integral (in the finitely additive case, of bounded functions vanishing outside a set of finite measure) corresponds to μ. A somewhat analogous situation prevails if there is an everywhere positive measurable function. The measurable sets correspond to the open and closed sets of a space (in the sense of A. D. Alexandroff). An unsolved question is whether or not there always exists an everywhere positive measurable function. (Received July 16, 1949.)

541*t*. Walter Rudin: *Integral representation of continuous functions.*

Let P denote a point in the finite plane domain D. Put $\Lambda F(P) = \lim_{r\to 0} 4(m(F; P, r) - F(P))/r^2$, where $m(F; P, r)$ is the mean of F on the circle of radius r about P. $\Lambda^* F$, $\Lambda_* F$ are defined likewise, with lim sup, lim inf in place of lim. Let Z be a compact set of capacity zero. Theorem I. Let $F(P)$ be continuous in D. Suppose (i) $\Lambda^* F(P) > -\infty$, $\Lambda_* F(P) < +\infty$ on $D - D \cdot Z$; (ii) there is a function $y(P)$, $y \in L$ on every closed subset of D, such that $y(P) \leqq \Lambda^* F(P)$ in D. Then $\Lambda F \in L$ on every closed subset of D, and, for almost all P in D, $F(P) = -(1/2\pi) \iint_R \Lambda F(Q)\, g(P, Q)\, dQ + H(P)$, where R is a bounded domain whose closure lies in D, and which contains P, $g(P, Q)$ is Green's function for R, and H is harmonic in R and assumes the boundary values of F. As a consequence, we obtain: Theorem II. Let $u(P)$ be continuous and subharmonic in D. Suppose $\Lambda_* u(P) < +\infty$ in $D - D \cdot Z$. Then, at all points P of D, $u(P) = -(1/2\pi) \iint_R \Lambda u(Q) g(P, Q) dQ + H(P)$, where H is the least harmonic majorant of u in R. In comparison with Riesz's theorem (Acta Math. vol. 54 (1930) p. 350), the significance of Theorem II lies in the appearance of a Lebesgue integral in place of a Stieltjes integral. Analogous results hold in spaces of higher dimension. (Received July 18, 1949.)

542. A. C. Schaeffer: *Fourier transforms of approximating functions.* Preliminary report.

Let $\phi_0(t)=c$, $\phi_1(t)$, $\phi_2(t)$, $\cdots$ denote a set of functions with continuous first order derivatives in a finite interval $0 \leqq t \leqq a$ such that every continuous function is the limit in the mean in this interval of finite sums $\sum a_\nu \phi_\nu(t)$. If $p(t)$ belongs to L_2 in $0 \leqq t \leqq a$ and its Fourier cosine transformation has exclusively real zeros, then there are finite sums $\sum a_\nu \phi_\nu(t)$ which converge in mean to $p(t)$ and whose Fourier cosine transformations have only real zeros. The smoothness conditions on the given sequence can be relaxed. The result is first proved in the special case in which the set ϕ_0, ϕ_1, ϕ_2, $\cdots$ is the set of functions $\cos(\nu\pi t/a)$, $\nu=0, 1, 2, \cdots$, and then in the general case using well known properties of entire functions of exponential type. (Received September 3, 1949.)

543*t*. H. M. Schaerf: *Determination of non-atomic measures.*

The family of all sets on which a measure m assumes a fixed value is called a *level family* of m. A class L of level families of m is termed *bounded* if there is a level family of m whose sets contain every set belonging to any family in L. A countably additive σ-finite measure m is called *non-atomic* if every set A of positive m-measure contains a measurable set B with $m(B)\cdot m(A-B)>0$. *Theorem: A non-atomic measure is uniquely determined (to within a multiplicative constant) by every infinite and bounded class of its level families.* The proof of this theorem relies on the Uniqueness Theorem in the abstract 544, *General theorems on the uniqueness and structure of invariant measures.* (Received July 14, 1949.)

544*t*. H. M. Schaerf: *General theorems on the uniqueness and structure of invariant measures.*

A measure m is called *invariant* under a relation R between some ordered pairs of measurable sets if ARB implies $m(A)=m(B)$. It is said to have the *intersection property* if, for every two sets A, B of positive measure, there are alike sets $A'\subset A$, $B'\subset B$ with $A'RB'$. R is called *regular* if it is an equivalence relation preserving countable disjoint partitions. *Uniqueness Theorem: In order that a σ-finite measure invariant under R be unique (to within a multiplicative constant) it is sufficient, and if R is regular also necessary, that every such measure have the intersection property. Structure Theorem: If a σ-finite measure m invariant under a regular relation R is unique, then any two sets of equal m-measure are almost congruent by countable partition. Moreover, then the values of m either are multiples of a positive number or fill a closed interval with left end point* 0. (Received July 14, 1949.)

545*t*. H. M. Schaerf: *Generalization of a theorem of Kakutani and Kodaira.*

This paper contains the proof of the following theorem: *Let G be an arbitrary locally compact topological group. Let S be the smallest μ-field of subsets of G which contains all closed compact G_δ-sets. Then the σ-field obtained by completing S with respect to any left-invariant measure defined in S and finite on some open set contains all closed compact subsets of G.* This theorem generalizes a result of S. Kakutani and K. Kodaira

(*Über das Haarsche Mass in der lokal bikompakten Gruppe*, Proc. Imp. Acad. Tokyo vol. 20 (1944) pp. 444–450). (Received July 14, 1949.)

546*t*. H. M. Schaerf: *On the uniqueness of Haar's measure.*

This paper removes all restrictions under which the uniqueness of *Haar's* measure is proved in the literature (like separability of the space or restriction to *Radon* measures). More generally, it contains the proof of the following theorem: *Let G be an arbitrary locally compact topological group. Let T be the smallest σ-field of subsets of G which contains all closed compact subsets of G. Let S be an arbitrary σ-field contained in T and containing a complete system of open neighborhoods of every point in G. Then a σ-finite left-invariant measure defined in S and finite on some open set is unique* (*to within a multiplicative constant*). (Received July 14, 1949.)

547*t*. H. M. Schaerf: *Uniqueness of an invariant measure in an abstract group.*

This paper contains a short and simple proof of the uniqueness of an invariant measure in an abstract group under more general assumptions than the ones made in A. Weil, *L'Intégration dans les groupes topologiques*, pp. 148–149. The proof is based on the Uniqueness Theorem of Abstract 544. (Received July 14, 1949.)

548*t*. Robert Schatten: *The space of completely continuous operators on a Hilbert space.*

In a Banach space whose elements are (some, not necessarily all) linear bounded operators on a Hilbert space, the norm of an element is in general different from its bound. In this connection a direct proof of the following theorem is given: Let $\mathfrak{C}$ denote the Banach space of all completely continuous (linear bounded) operators on a Hilbert space $\mathfrak{H}$, where the norm of an operator is given by its bound. Then, its first adjoint space $\mathfrak{C}^*$ may be interpreted as the trace-class. The trace-class is the Banach space of all linear bounded operators A on $\mathfrak{H}$, for which $\sum_i((A^*A)^{1/2}\phi_i, \phi_i) < +\infty$ for a complete orthonormal set (ϕ_i); the last sum which is independent on the chosen complete orthonormal set represents the norm of A in $\mathfrak{C}^*$. Moreover, the second adjoint space $\mathfrak{C}^{**}$ of $\mathfrak{C}$ may be interpreted as the Banach space of all linear bounded operators on $\mathfrak{H}$ where again the bound of an operator represents its norm. (Received June 30, 1949.)

549*t*. Annette Sinclair: *Generalization of Runge's theorem to approximation by analytic functions.*

Runge's Theorem on approximation of a function analytic on a closed set by a rational function is generalized to approximation on a set S whose components are closed and whose sequential limit points are in the complement of S. (A sequential limit—s.l.—point is a limit point of some set consisting of one point of each component of S.) The approximating function is analytic except possibly on a preassigned set C consisting of (1) the s.l. points and (2) one point of each component of the complement of S which contains no s.l. point. If the restriction is placed on the function to be approximated that it be nonvanishing on S and that its logarithm be single-valued on S, the approximating function can be required to have not only its singularities but also its zeros in C. The approximation obtained is not only uniform on S but is such that the closeness of approximation can be preassigned independently for each component. (Received July 5, 1949.)

550. H. L. Smith: *Relative contingent and n-hedra to paths.*

If P_0 is a limit point of a set E in a Euclidean space and M is a class of vectors in that space, then a vector ξ_0 is a semi-tangent to E at P_0 relative to M if there is in E a sequence of points $\{P_n\}$ which converges to P_0 and is such that sig $\omega(P_n-P_0, M)$ converges to ξ_0, where $\omega(P_n-P_0, M)$ denotes the component of the vector P_n-P_0 which is orthogonal to M, and sig $\omega(P_n-P_0, M)$ is $\omega(P_n-P_0, M)$ divided by its length if $\omega(P_n-P_0, M)\neq 0$ and is 0 otherwise. The contingent to E at P_0 relative to M is the set of all its semi-tangents at P_0 relative to M. This concept is used to give an intrinsic definition of n-hedron to a path. Conditions necessary and conditions sufficient for the existence of such an n-hedron are derived. (Received July 20, 1949.)

551. E. J. Specht: *Estimates on the mapping function and its derivatives in conformal mapping of nearly circular regions.*

Let C denote a simple, closed curve with polar equation $\rho=\rho(\phi)$, $0\leqq\phi\leqq 2\pi$, where $\rho(\phi)$ is positive, continuous, and periodic of period 2π. Let the function $w=f(z)$ map the circle $|z|<1$ conformally onto the interior of C in such a manner that $f(0)=0$ and $f'(0)>0$. If, for some positive ϵ, $1\leqq\rho(\phi)\leqq 1+\epsilon$ and if there is a positive number M such that $|\rho(\phi+\delta)-\rho(\phi)|\leqq M\epsilon\delta$ where δ is any real number, then there exists a constant K which depends only on M such that $|f(z)-z|\leqq K\epsilon$. In fact $K\leqq 1+M(2\log 2+1)$ if $\epsilon\leqq 1$. (This result was stated by A. R. Marchenko in 1935 without proof and without a numerical estimate of K.) By imposing further restrictions on the curve C (one of which involves essentially the tangent angle or the curvature), analogous numerical estimates of $|(d/d\theta)\arg f(e^{i\theta})-1|$, $|f'(z)-1|$, $|(d^2/d\theta^2)\arg f(e^{i\theta})|$, and $|f''(z)|$ are obtained and the existence of bounds for $|(d^n/d\theta^n)\arg f(e^{i\theta})|$ and $|f^{(n)}(z)|$ $(n\geqq 3)$ is established. The principal tools used in the proofs are the Poisson integral representation of a harmonic function in the unit circle and the well known formula for the conjugate $\bar{g}(\theta)$ of an integrable function $g(\theta)$. (Received July 18, 1949.)

552. Otto Szasz: *On the Gibbs' phenomenon for Euler means.*

The classical Gibbs' phenomenon for the partial sums of certain Fourier series at a point of discontinuity is here generalized to Euler means. It is shown that the sequence of Euler's means presents essentially the same Gibbs' phenomenon. Similar results hold for certain Hausdorff means and for Borel's summability method. (Received July 12, 1949.)

553. H. P. Thielman: *On functional equations and means.*

Some of the results of this paper are generalizations of the results of Aczél (Comment. Math. Helv. vol. 21, pp. 247–252). The method for deriving the results consists in reducing the problems to the solution of the well known Cauchy functional equations. Functional equations of the form $F(ax+by+c)=\alpha G(x)+\beta H(y)+\gamma$ are solved and the results are used to derive results for generalized means. Solutions to sets of simultaneous functional equations are also found. These functional equations are broad generalizations of such functional equations as $g(xy)=y^\beta g(x)+x^\beta g(y)$, which appeared recently in the mathematical literature (Amer. Math. Monthly vol. 56 (1949) p. 414). (Received July 15, 1949.)

554. J. L. Ullman: *A problem of Tchebychef.* Preliminary report.

Tchebychef proposed, as a method of approximating an integral $\int_0^1 f(t)dt$, the formula $(1/n)\sum_1^n f(t_{ni})$, where, for each n, $n=1, 2, \cdots$, the n numbers $t_{n1}, \cdots, t_{nn}$ satisfy the condition that $\int_0^1 P_n(t)dt=(1/n)\sum_1^n P_n(t_{ni})$, where $P_n(t)$ represents an arbitrary polynomial of degree n. S. Bernstein showed that for sufficiently large n, the numbers t_{ni} would not be real, so that this method would not be applicable if $f(t)$ were defined for real values only. This suggests the following problem. Let $\alpha(t)$ be a positive, continuous function on the open interval $(-1<t<1)$ satisfying $\int_{-1}^1\alpha(t)dt=1$. As an approximation formula for the integral $\int_{-1}^1 f(t)\alpha(t)dt$, consider the expression $(1/n)\sum_1^n f(t_{ni})$, where for each n, the numbers $t_{ni}, \cdots, t_{nn}$ satisfy the condition that $\int_{-1}^1 P_n(t)\alpha(t)dt=(1/n)\sum_1^n P_n(t_{ni})$, where $P_n(t)$ is an arbitrary polynomial of degree n. For which functions $\alpha(t)$ will the limit points of the set t_{ni} $(i=1, \cdots, n; n=1, 2, \cdots)$ all lie on the interval $(-1\leqq t\leqq 1)$? For those $\alpha(t)$ for which this is true, the integral can be approximated by the sum for any $f(t)$ analytic in a domain containing the interval $(-1\leqq t\leqq 1)$. It is a well known result, that if $\alpha(t)$ is given by $(1/\pi)(1-t^2)^{-1}$, then the t_{ni} will be the roots of the Tchebychef polynomials of the interval $(-1\leqq t\leqq 1)$ and will all lie in this interval. The more general result obtained, using theorems on the derived set of zeros of Faber polynomials, is that if $\alpha(t)=(1/\pi)(1-t^2)^{-1/2}\operatorname{Re} zF'(z)/F(z)$, $z=t+(t^2-1)^{1/2}$, where $F(z)$ is the normalized exterior mapping function of a simple analytic curve, symmetric to the real axis, and contained in the circle $|z|\leqq 1$, then the limit points of the t_{ni} will all lie on the interval $(-1\leqq t\leqq 1)$. (Received July 18, 1949.)

555t. Antoni Zygmund: *A remark on functions of several complex variables.*

Let $f(z_1, z_2, \cdots, z_k)$ be a function of complex variables $z_1, \cdots, z_k$ regular for $|z_1|^2+\cdots+|z_k|^2<1$, and such that the integral $\int_{\sigma_r}\log^+|f|d\sigma_r$ is bounded for $r<1$-where σ_r denotes the sphere $z_1^2+\cdots+z_k^2=r^2$ and $d\sigma_r$ the element of the $((2k-1)$, dimensional) volume of the sphere. Then f has a finite non-tangential limit at almost every point of the sphere $|z_1|^2+\cdots+|z_k|^2=1$. As a corollary of this and of a recent result of Rauch, one obtains that if $\int_{\sigma_r}|f|^p d\sigma_r$ is bounded for some $p>0$, then $\int_{\sigma_r}|f(z_1, \cdots, z_k)-f(rz_1, \cdots, rz_k)|^p d\sigma_r\to 0$ as $r\to 1$. (Received August 2, 1949.)

Applied Mathematics

556. Nathaniel Coburn: *"Characteristic directions" in three-dimensional supersonic flows.*

It has been shown (*Supersonic flow and shock waves*, R. Courant and K. O. Friedrichs, Interscience, New York, 1948, pp. 71–75; *Nonlinear hyperbolic differential equations for functions of two independent variables*, K. O. Friedrichs, Amer. J. Math. vol. 70 (1948) pp. 555–589) that a system of n-totally hyperbolic quasilinear first order partial differential equations can be replaced by another system (consisting of linear combinations of the original equations) with the property that each of the latter equations involves differentiation in only one direction. Such directions are called "characteristic directions." In the present paper, it is shown that no such "characteristic directions" exist for non-isentropic or isentropic, irrotational, steady, three-dimensional flows. If "characteristic pairs of directions" are defined by the condition that the linear combinations of the original equations involve differentiations with respect to two directions, then such directions always exist for isentropic, irrotational flows. In fact, any two directions lying in the planes which envelope the bicharacteristic cone are permissible. (Received February 11, 1949.)

557. K. L. Conrad: *Stress distribution due to hydrostatic pressure on a parabolic boundary.*

The boundary condition for the stress distribution due to hydrostatic pressure on a parabolic boundary has been expressed in parabolic coordinates as a single integral by means of the Fourier integral theorem. A biharmonic solution, X, can be expressed in the terms of two complex functions $\phi(z)$ and $f(z)$ by means of the expression $X=\bar{z}\phi(z)+z\bar{\phi}(\bar{z})+f(z)+\bar{f}(\bar{z})$. The two relationships of the stresses in parabolic coordinates, $J_{\xi\xi}+J_{\eta\eta}$ and $J_{\xi\xi}-J_{\eta\eta}+2iJ_{\xi\eta}$, are determined by the complex functions $\phi(z)$ and $f(z)$. A finite form has been obtained for those complex functions from which the stresses and displacements have been determined. (Received July 16, 1949.)

558. M. Dresden: *The Birkhoff "billiard ball problem" in quantum mechanics.*

Many of the problems of dynamics can be formulated in a natural fashion in quantum mechanics. In particular those problems dealing with the existence and explicit evaluation of time averages would appear to find a more appropriate formulation in terms of quantum theory. This reformulation has been performed for the Birkhoff "billiard ball problem." In some cases one can obtain explicit solutions to the quantum mechanical problem. Then one can utilize the connection between the classical time averages and the approximate (WKB) quantum averages to evaluate explicitly some classical averages. This general connection suggests two types of problems. The first group of problems concerns the relationship between the results in classical mechanics of Poincaré and Birkhoff and similar ones in quantum mechanics. The second group of problems refers to the fashion in which these classical results are contained in the quantum mechanical formalism. These general problems will be discussed using the "billiard ball problem" as an example. (Received July 15, 1949.)

559. R. N. Goss: *Center of flexure of beams of triangular section.*

The center of flexure is that point of loading in a section of a beam for which the local twist vanishes at the centroid of the section. For a beam of right-triangular section with vertices at $(0, 0)$, $(0, c)$, $(-c/m, 0)$ in the plane $z=z_1$, when the load acts parallel to the hypotenuse, the center of flexure is found to lie on the line $y-mx=[c(3+4\sigma)/5(1+\sigma)]-[4\sigma Dm(m^2+1)/\mu c^3(1+\sigma)]$ in this plane, where σ is Poisson's ratio, μ is the modulus of rigidity, and D is the torsional rigidity. In the case of the general triangular section with sides $y=d$, $y=m_1x$, $y=m_2x$ and any direction of load, the coordinates of the center of flexure have been found for $\sigma=1/2$. The results agree with those obtained previously for the isosceles-triangular section (Bull. Amer. Math. Soc. Abstract 55-7-412). (Received July 14, 1949.)

560. Eberhard Hopf: *Integration of Burgers' simplest space-time model for free turbulence.*

As a simplified model for turbulent fluid flow not activated by external forces (free turbulence), J. M. Burgers considered the partial differential equation $u_t+uu_x=\nu u_{xx}$ (see his article in *Advances in Applied Mechanics*, edited by v. Mises and v. Kármán, New York, 1948, pp. 171–196, in particular pp. 182–192). The present author supplements Burgers' approximative study of his equation by a rigorous and explicit solution of the problem. (Received July 14, 1949.)

561t. Robert Kahal: *The realization of the transfer function of a four-terminal network.*

The transfer function, $\theta(p)$, of a four-terminal network is defined as the ratio of the output voltage to the input voltage. The following theorems are proved. (I) The necessary and sufficient conditions that a function $\theta_L(p)$ shall be the transfer function of an L network are: (1) $\theta_L(p)$ has no poles or zeros in Re $(p)>0$. Poles on the boundary are simple. (2) $\theta_L(p)$ is real when p is real, and $0<\theta_L(p)<1$ when p is real and positive. (3) $|\arg \theta_L(p)|\leqq\pi$ when $|\arg p|=\pi/2$. (II) The necessary and sufficient conditions that $\theta(p)$ shall be, to within a constant factor, the transfer function of a ladder network are: (1) $\theta(p)$ has no poles or zeros in $\mathrm{Re}(p)>0$. Poles on the boundary are simple. (2) $\theta(p)$ is real when p is real, and $0<\theta(p)<1$ when p is real and positive. (III) The necessary and sufficient conditions that a function $\psi(p)$ shall be the product of two positive real functions are: (1) $\psi(p)$ has no poles or zeros in $\mathrm{Re}(p)>0$. (2) $|\arg \psi(p)|\leqq\pi$ when $|\arg p|=\pi/2$. (3) $\psi(p)$ is real when p is real. A method for the synthesis of the L and ladder networks is given. (Received July 16, 1949.)

562. M. H. Martin: *Steady, plane, rotational Prandtl-Meyer flows of a polytropic gas.*

The theory of Prandtl-Meyer flows, first discovered among the irrotational flows of a polytropic gas, is extended to rotational flows of a polytropic gas. If the streamlines ($\psi=$const.) and the isobars ($p=$const.) are taken as curvalinear coordinates, the general problem of determining steady plane flows can be reduced to finding two functions, $q=q(p, \psi)$, $\theta=\theta(p, \psi)$ (that is, the speed and direction of flow) so that they satisfy the equation $q((q_{pp}-q\theta_p^2)/\theta_\psi)_\psi+(q^2\theta_p)_p=0$. If $q=q(p,\psi)$ is preassigned, the problem of determining Prandtl-Meyer flows reduces to the determination of solutions of this equation having the form $\theta=\theta(q)$, in which case the equation reduces to an ordinary differential equation of Bernoulli type. The properties of rotational Prandtl-Meyer flows are investigated with the aid of the solutions of this ordinary differential equation. (Received July 8, 1949.)

563. P. F. Nemenyi and A. H. Van Tuyl: *Two-dimensional plastic stress systems with isometric principal stress trajectories.*

Carathéodory and Schmidt, in their paper on the maximum shear trajectories in plane plastic stress with the yield condition $\tau=(\sigma_1-\sigma_2)/2=$constant, discussed the possibility of these trajectories forming an isometric net. They found that the only such isometric nets are characterized by the complex function $Z(z)$ satisfying the equation $z+a_1+ia_2=\int \exp\,[aZ^2+(b_1+ib_2)Z+c_1+ic_2]dZ$, where a, a_1, a_2, b_1, b_2, c_1, and c_2 are real constants. The same nets are also possible principal stress trajectories. The authors generalize the inquiry of Carathéodory and Schmidt to an arbitrary yield function $\tau=f(\sigma)$ $(2\sigma=\sigma_1+\sigma_2)$. When the principal stress trajectories form an isometric net corresponding to the function $Z=\phi+i\psi$, the Lamé equations of equilibrium give the relations $2\log|dZ/dz|=\log f(\sigma)+F(\phi)-G(\psi)$ and $\int d\sigma/f(\sigma)=F(\phi)$ $G(\psi)$. A complete discussion of these equations shows that only for four distinct families of yield conditions, depending on from two to five parameters, do there exist isometric nets satisfying the conditions of the problem beyond the trivial nets consisting of concentric circles and concurrent straight lines. The functions Z belonging to each family are given either explicitly or in the form of quadratures. It is found that any linear yield function is consistent with the same family of isometric nets as the yield function τ = constant. (Received September 12, 1949.)

564. L. E. Payne: *Torsion and flexure of composite sections. I. Torsion.*

The Saint Venant torsion problem for composite cylinders of various types of cross sections has been considered. The sections investigated fall into two general groups. In the first group are sections made up of two different isotropic materials. The corresponding solutions are obtained for the cases of eccentric circles, confocal and concentric ellipses, as well as rectangular sections. In the second group the sections are partly isotropic and partly orthotropic. Here, solutions are obtained for the concentric circular and confocal elliptic sections and the rectangular section. It is shown that in the first group the rigidity of the composite section may be obtained by adding together the rigidities of the two different isotropic parts only if the lines dividing them are lines of shearing stress. The rigidities in the various cases are compared with those corresponding to the completely isotropic and completely orthotropic sections. The flexure problem is currently being investigated. (Received July 12, 1949.)

565. B. R. Seth: *Finite elasto-plastic torsion.*

In the customary treatment of elasto-plastic torsion the displacements and the strains are both assumed to be small. In a number of plastic problems both these assumptions need not hold good. We propose to discuss the torsion of a circular cylinder by taking finite components of displacements and strains. The problem is examined from the point of view of both the rival theories of plasticity—the theory of plastic flow and the theory of plastic deformation. For a first approximation in which axial stresses are neglected both the theories give the same results. A second approximation has been carried out and the corresponding torsional couple and stresses have been calculated. (Received July 11, 1949.)

566. Domina E. Spencer: *Separability conditions for some equations of mathematical physics.*

Though considerable research has been done on separability conditions for the equations of mathematical physics, no comprehensive treatment of the subject is available. In this paper, the method of Robertson (Math. Ann. vol. 98 (1927) p. 749), which was originally applied to simple separability of the Schrödinger equation, is extended to obtain necessary and sufficient conditions for both simple separability, $\phi = U^1(u^1)U^2(u^2)U^3(u^3)$, and R-separability, $\phi = U^1(u^1)U^2(u^2)U^3(u^3)/R(u^1, u^2, u^3)$, of the Laplace, Helmholtz, diffusion, wave, and Schrödinger equations. The conditions are expressed in terms of Stäckel matrices. (Received July 11, 1946.)

567t. John Todd: *The condition of certain matrices.*

Two measures (the N-condition number and the M-condition number) for the condition of a matrix A have been suggested by A. M. Turing (Quarterly Journal of Mechanics and Applied Mathematics vol. 1 (1948) pp. 287–308). A third measure is suggested by the results of John von Neumann and H. H. Goldstine (Bull. Amer. Math. Soc. vol. 53 (1947) pp. 1021–1099); we call $|\lambda|/|\mu|$ the P-condition number of A if $|\lambda|$, $|\mu|$ are the greatest and least of the absolute values of its characteristic roots. These measures (or reasonable appraisals of them) have been calculated for the matrices which arise in the approximate solution of ordinary and partial differential equations. Elementary methods, Rayleigh's Principle, and results of D. E. Rutherford (Proc. Royal Soc. Edinburgh vol. A52 (1947) pp. 229–236), among others, are employed. The results obtained show that these measures are in substantial agreement among themselves and with the experience of the practical computer (L. Fox, Proc. Cambridge Philos. Soc. vol. 45 (1949) pp. 50–68). Among the results is confirmation of the observed fact that the matrices occurring in the approximate solution of a partial differential equation, for example, $z_{xx}+z_{yy}=kz$, are in better condition than those arising from the corresponding ordinary differential equation, for example, $y''=ky$. (Received July 14, 1949.)

568. C. A. Truesdell: *On finite strain of an elastic body.*

There have been numerous recent investigations of the general theory of elasticity. For the case when the work of deformation is completely stored as strain energy, however, the general theory has been complete for a long time. Both Eulerian and Lagrangian strain measures were defined by Cauchy (1823, 1827, 1841). Green (1839) introduced the strain energy in full generality, and Kirchhoff (1852) in deriving a second order theory gave formal apparatus and methods from which the fully general theory follows easily, although it was not actually worked out until later by Kelvin (1863) and Boussinesq (1872). Finger (1894) discovered that for isotropic bodies the strain energy may be regarded as a function of Eulerian strain measures only, and worked out the resulting simplified form of the stress-strain relations. Apparently one reason for so many recent re-discoveries of classical results is the wilderness of long and awkward notations still in vogue. Rivlin (1948) pointed out that for incompressible materials the stresses are determined by the energy equation only up to an arbitrary Eulerian hydrostatic pressure, and proposed a simple one modulus nonlinear theory. In the present note Rivlin's approximate theory is compared with that of Kirchhoff. It is shown that Rivlin's theory results from discarding all terms retained by Kirchhoff but retaining the first order term, representing an initial Lagrangian pressure, which Kirchhoff put equal to zero. Since it is the arbitrary Eulerian pressure resulting from the condition of incompressibility which makes the resulting theory nontrivial, a corresponding approximation for compressible materials could not lead to useful results. The Rivlin theory can thus be put in a symmetrical form: the stress is always the sum of a Lagrangian pressure and an Eulerian pressure. (Received July 16, 1949.)

GEOMETRY

569. N. A. Court: *On the polar circles of the faces of a tetrahedron.*

Let H_a be the orthocenter of the face BCD of a tetrahedron $(T)=ABCD$, (H_a) the sphere having for great circle the polar circle of the triangle BCD, and (P_a) the orthoptic sphere of (H_a). Analogous elements are attached to the other faces of (T). Let (H'') denote the tetrahedron having for vertices the orthocenters H_a, H_b, $\cdots$. a. The sum of the squares of the distances of the circumcenter of (T) to the orthocenters of its faces is equal to twice the sum of the squares of the radii of the polar circles of the faces of (T) increased by the square of the circumdiameter of (T). b. The orthogonal sphere of the four spheres (H_a), (H_b), $\cdots$ is coaxal with the circumspheres of the tetrahedrons (T) and (H''). c. The circumsphere of (T) is orthogonal to the six spheres of similitude of the four spheres (H_a), (H_b), $\cdots$ taken in pairs. d. The orthogonal sphere of the four spheres (P_a), (P_b), $\cdots$ coincides with the circumsphere of the tetrahedron (T). (Received July 15, 1949.)

570. V. G. Grove: *A note on isothermal nets.*

This paper gives a new characterization of isothermally conjugate nets using a notion we have called the planar property of a family of congruences associated with a curve of the net. The net is isothermally conjugate if and only if any one of these congruences has the planar property. An extension is made to isothermally orthogonal non-conjugate nets. A by-product of the paper is a method of generating all classical canonical lines. (Received June 15, 1949.)

571. W. R. Hutcherson: *A cyclic involution of order eleven.*

In two previous papers (Bull. Amer. Math. Soc. vol. 37 (1931) pp. 759–765; and vol. 40 (1934) pp. 143–151) cyclic involutions of orders three, five, and seven were discussed. This paper considers the transformation T, $x_1' : x_2' : x_3' : x_4' = x_1 : \epsilon x_2 : \epsilon^2 x_3 : \epsilon^3 x_4$ where $\epsilon'' = 1$. C is defined as any curve on the quartic surface, F_4, $ax_2x_3^3 + bx_1x_2x_4^2 + cx_1x_3^2x_4 + dx_2^2x_3x_4 = 0$, passing through $P_3(0010)$, a point of coincidence (invariant point). The image of C under T is another curve on F_4 passing through P_3. This point P_3 is called "a perfect point of coincidence" if C and every image of C touches the same line at P_3 for every tangent to F_4 at P_3. However, P_3 is found to be a non-perfect point. Three other non-perfect points are $P_1(1000)$, $P_2(0100)$, and $P_4(0001)$. It was known that, for an involution of order three, points in the first order neighborhood of P_3 along the two invariant directions are perfect. For I_5 (involution of order five) one may have second order neighborhood perfect points as well as first order; for I_7, third order as well as second order. This paper found that P_3, which is simple on F_4, has a perfect point in the third order neighborhood along the $x_1 = x_2 = 0$ direction while along the other invariant direction $x_2 = x_4 = 0$, one must go out to the point of the fourth order neighborhood. The involution contained on F_4 was mapped upon a ϕ surface of order 44 in a space of thirty-two dimensions. (Received July 14, 1949.)

572*t*. B. E. Mitchell: *The complex complete quadrangle and quadrilateral.*

This paper is presented in the interest of constructions in the complex plane. The complete quadrangle, tetrastigm (Lachlan), is a configuration consisting of seven points and nine lines. The complete quadrilateral, tetragram, is made up of seven lines and nine points. The two configurations are dual. Interest centers in the distribution of the real and the imaginary among the sixteen elements of each configuration as step by step the base elements, four points in the one case and four lines in the other, become imaginary. The case in which all the base points (lines) become imaginary and conjugate in pairs receives special attention. In this case half the elements are real and half are imaginary—ideal complexity. All imaginary elements, both points and lines, are given real representation. (Received July 10, 1949.)

573*t*. T. S. Motzkin: *The lines and planes connecting the points of a finite set.*

The author's conjecture (Dissertation, Basel, Jerusalem 1936, p. 31) that, in d-dimensional projective space over an arbitrary coördinate field, n points that are not on one hyerplane determine at least n hyperplanes is proved by use of a combinatorial lemma and an idea of de Bruijn and Erdös, Indagationes Mathematicae vol. 10 (1948) p. 422, who proved the case $d = 2$. For the real plane the theorem follows also from Sylvester's conjecture, proved by Gallai, that if n points are not on one straight line there exists an ordinary line, that is, a straight line containing only two of the points. The question of de Bruijn and Erdös whether the number of ordinary lines tends to infinity with n is answered in the affirmative. It is further shown that n points in real 3-space that are not on one plane determine at least four ordinary planes, that is, planes on which all of the given points but one are collinear. For general coördinate fields and $n=7$ or $n>8$ no ordinary line need exist. (Received July 22, 1949.)

Logic and Foundations

574*t*. A. R. Schweitzer: *An examination of Bergson's "Creative evolution."*

Bergson's *Creative evolution* is examined with regard to: I. doctrine of opposites, II. main working hypothesis, III. theory of relations, IV. references to orientation. The author interprets Bergson's treatise as a theory of knowledge, primarily of biological reference, elaborating a discrimination between "intuition" and "intellect." For intuition is associated by Bergson with instinct, life, the vital order, creation, invention, non-mathematical time, duration; and intellect is associated with intelligence, matter, the geometrical order, logic, space. The preceding two opposed sets of terms permit transition to one of Bergson's central themes, namely, his working hypothesis of divergent lines of evolution mutually complementary (inverse) and motivated by a common impulse (impetus). An analogy is found between Bergson's working hypothesis and the author's "principle of furcation" reported in Bull. Amer. Math. Soc. vol. 21 (1915) pp. 376–377, abstract 5; vol. 22 (1916) p. 294, abstract 25; see also ibid. vol. 20 (1914) p. 454, abstract 8. Relations are associated by Bergson with intelligence whose function it is to establish them; relation is essentially comparison. Bergson's references to orientation include ascent, descent, motion (movement) organization, order, inversion, complementariness, oppositeness. (Received July 18, 1949.)

575*t*. A. R. Schweitzer: *On a relation of mathematics to biology.*

This paper aims to effect a gradual transition from mathematics to biology based on the author's foundations of geometry in terms of "right" and "left" (sameness and difference of orientation) in Amer. J. Math., 1909, 1913 with the monograph of W. Ludwig, *Das Rechts-Links-Problem im Tierreich und beim Menschen* (Berlin, 1932) and Kant's philosophy as connecting bonds. The author cites Ludwig, loc. cit., p. 442, paragraphs 2, 3. Ludwig asserts "right" and "left" to have been the source of Kant's philosophy and, more generally, to have been associated of old with pairs of opposites (for example, the author notes, by the Pythagorean school of philosophy). The author remarks that "right" and "left" do not seem to occur explicitly in Kant's *Critique of pure reason*; however, the latter treatise abounds in pairs of opposites. Reference is made to the statement of Sir James Jeans (*Physics and philosophy*, Cambridge and New York, 1946, p. 38) that Kant was one of the earliest of biological evolutionists. (Received July 18, 1949.)

576. D. M. Studley: *Identity, equivalence, and distinction.*

The three elements of conception, identity, equivalence, and distinction, are considered with attention centered upon relations among them. In order to obtain full grasp on the relations, the system of method of the author's Lawrence paper (*The mathematics of language*) is used. Individual point or element identity is separated in thought from identity of form or class. Equivalence which is contained in identity is shown to be intersection of form. Further, equivalence is defined as the space between identity and distinction. The third element, distinction, is seen as the sum of two identities modulo two. Corporeal distinction is separated in thought from distinction of form. The author's concept of abstract isomorphism (Mathematics Magazine vol. 22 (1949) pp. 191–193) is classed as equivalence. A clarification of orientation results from this study. (Received June 2, 1949.)

577. Wanda Szmielew and Alfred Tarski: *Theorems common to all complete and axiomatizable theories*. Preliminary report.

All the theories considered are formalized within lower predicate calculus; they are supposed, for simplicity, to contain one non-logical constant—operation symbol $+$. Consistent, complete, axiomatizable (that is, with a finite system of non-logical axioms) theories of this kind will be called *CA-theories*. Many different examples of CA-theories are known; for example, the theories of (i) groups of a given prime order, (ii) atomless Boolean algebras, (iii) simply and densely ordered lattices with zero and unit. Nevertheless, *there are non-tautological statements derivable from axioms of every CA-theory*; they will be called *CA-statements*. For example, the negated conjunction of all non-logical axioms of any axiomatizable, essentially undecidable theory (see Journal of Symbolic Logic vol. 14, p. 75) is a CA-statement. Results in Bull. Amer. Math. Soc. Abstract 55-1-79 lead to another class of CA-statements. It consists of the statements which sometimes fail when applied to group operations in infinite commutative groups, but hold for any other operation $+$; for example: *If, for all x, y, z, $x+(y+z)=(x+z)+y$, and $x=y+(u+u)$ for some u, then, for no x, $(x+x)+x=x \neq x+x$* (in other words, if $+$ is a commutative group operation with all elements divisible by 2, then no element is of order 2). Finally notice: there are consistent, complete, non-axiomatizable theories in which all the CA-statements hold. (Received June 27, 1949.)

Statistics and Probability

578*t*. Maria Castellani: *Theorems on convergence of compound distributions with symmetric components.*

The theorems deal with operations of convolution in R_1 concerning specific families of distributions. Thus, $K_\alpha(x)=F(x)*G_\alpha(x)$ is obtained by combining any d.f. $F(x)$ with any d.f. $G_\alpha(x)$ which depends on a parameter under the restriction of symmetry when $G_\alpha(x+h)+G_\alpha(x-h)=1$ for any $h>0$. A generalization of Cantelli's inequalities will enable us to prove that a given d.f. $F(x)$ in R_1 can by association with a convenient symmetric d.f. $G_\alpha(x)$ of continuous type give a continuous d.f. $K_\alpha(x)$, which for $\alpha\to\infty$ converges asymptotically almost everywhere to the given $F(x)$. Therefore at any continuity bordered interval for a given $F(x)$ there exist convenient, uniformly convergent series of continuous functions which asymptotically approach the given d.f. $F(x)$. (Received July 14, 1949.)

579*t*. Herman Chernoff: *Remarks on a rational selection of a decision function.*

Problems in the theory of statistics seem to reduce to the selection of a strategy (decision function) from several available strategies. The problem of the selection of such a strategy is analogous to that in the two person zero sum game where nature is the statistician's opponent. A major difference between these two problems lies in the fact that it does not seem proper to regard the unknown state of nature as a strategy of an opponent. Several criteria previously mentioned for determining a strategy are discussed and seem to have inadequacies. An axiom system is constructed for a criterion to be considered "rational." When applied to the set of problems where there are only two states of nature, the results are that the only criterion satisfying the axiom system is equivalent to assuming an a priori probability of 1/2 for each state of nature. (Received July 14, 1949.)

580*t*. K. L. Chung and Paul Erdös: *Probability limit theorems assuming only the first moment.* Preliminary report.

We consider independent random variables $X_1, X_2, \cdots$, having the same distribution $F(x)$; $S_n = \sum_{k=1}^{k=n} X_k$. The novelty consists in that about this distribution we assume only $\int_{-\infty}^{\infty} x dF(x) = 0$ or both $\int_0^{\infty} x dF(x)$ and $\int_{-\infty}^{0} x dF(x)$ diverge. Analytical and combinatorial methods are developed for treating this case, the latter being more successful at present. For simplicity we state results for the case where each X assumes only integral values and such that if a is any integer, $P(S_n = a) > 0$ for sufficiently large n. Estimates of $P(S_n = a)$ are obtained; in particular we prove that $\lim P(S_n = a)/P(S_n = b) = 1$. Further let $N_n(a)$ and $N_n(b)$ denote the number of a's and b's respectively among the sequence $S_1, \cdots, S_n$, then $P(\lim N_n(a)/N_n(b) = 1) = 1$. Extensions to continuous distributions will be considered. (Received July 18, 1949.)

581. C. L. Dolph and M. A. Woodbury: *Optimal linear prediction of stochastic processes whose covariances are Green's functions.*

A method of unbiased, minimal variance, linear prediction is developed for problems similar to those of prediction and filtering treated by Wiener. It differs from these in that the unbiased condition is imposed, only a finite part of the past is employed, and no stationary assumption is used. It is shown that the special stationary case discussed by Cunningham and Hund, *Random processes in problems of air warfare* (Supplement to the Journal of the Royal Statistical Society, 1946) succeeds because the correlation function $e^{-\lambda|t-s|}$, well known to that of the process defined by the Langevian equation, is the Green's function of the homogeneous differential equation formed by letting the adjoint differential operator of the Langevian equation operate on the operator of this equation. This relationship is shown to persist for any physically stable linear differential equation driven by "white noise." The well known equivalence between integral and differential equations is then extended by use of Stieltjes integrals and used to effect the solutions of the integral equations of the first kind which yield the "optimum" linear prediction. The nonstationary example consisting of purely random motion about a mean linear path in the presence of radar type errors is treated in detail. (Received June 27, 1949.)

582. H. H. Germond: *The integral of the Gaussian distribution over the area bounded by an ellipse.*

This paper describes the preparation of tables from which to obtain the integral of a bivariate Gaussian distribution over the area of an ellipse. The center of the ellipse need not coincide with the mean of the Gaussian distribution, nor need the axes of the ellipse have any special orientation with respect to the Gaussian distribution. (Received July 15, 1949.)

583. P. G. Hoel; *On the problem of optimum classification.*

Let f_i, $i = 1, 2, \cdots, k$, be the probability density function of population i and let p_i be the probability that population i will be sampled. Assume certain differentiability conditions and moment properties. Then, for known parameters, the probability of a correct classification will be maximized by choosing the region M_i, which corresponds to classifying into population i, as that part of variable space where $p_i f_i \geqq p_j f_j$, $j=1, 2, \cdots, k$. If the parameters are unknown, an asymptotically optimum set of estimates will be given by the set that minimizes a certain form in the covariances. Among uncorrelated estimates, maximum likelihood estimates are seen to be asymptotically optimum. If weight functions, W_{ij}, are introduced and the expected value of the loss is minimized, the same methods of proof show that the region M_i becomes that part of variable space where $\sum_{r=1}^{k} p_r f_r (W_{rj} - W_{ri}) \geqq 0$, $j=1, 2, \cdots, k$, and that the criterion for an asymptotically optimum set of estimates is of the same form as the preceding criterion. (Received July 11, 1949.)

584. Leo Katz: *On the relative efficiencies of BAN estimates.*

J. Neyman, in the *Proceedings of the Berkeley Symposium on Mathematical Statistics and Probability*, 1949, proved that χ^2 minimum estimates with either of two alternative definitions of χ^2 are efficient, as also are the maximum likelihood estimates. He also raised the question whether some of these estimates were better than others. This paper bears on that question. In making χ^2 minimum estimates, it is often necessary to avoid small frequencies by grouping together at least one tail of the distribution. It is with respect to the parameters of these *modified* distributions that the χ^2 estimates are efficient. Define *relative efficiency* in these circumstances as the ratio of the variance of an efficient estimator in the unmodified case to that of one in the modified case. It is shown that, except for a rectangular probability law, the relative efficiency <1 and, further, it decreases as the tail grouping is made wider. Formulae are given for the relative efficiencies of χ^2 minimum estimators for binomial and Poisson probability laws and some representative values computed to exhibit these effects. (Received July 18, 1949.)

585*t*. Julius Lieblein: *Partial sums of the negative binomial in terms of the incomplete beta-function.*

In acceptance sampling a certain size sample is taken at random from a lot of items and the lot is accepted if the number of defective items do not exceed a predetermined number characteristic of the sampling plan. The Statistical Engineering Laboratory has been studying the probabilities that a decision to accept or reject can be made before the sample is completely inspected. Such probabilities are found to involve certain sums apparently not previously treated. In this note the author proves a simple identity connecting these sums which greatly facilitates their computation and shows how they may be written in terms of the well known incomplete beta-function of Karl Pearson, for which extensive tables are available. (Received July 14, 1949.)

586. Jack Sherman and Winifred J. Morrison: *Adjustment of an inverse matrix corresponding to changes in the elements of a given column or a given row of the original matrix.*

A simple computational procedure is derived for obtaining the elements b'_{ij} of an nth order matrix (B') which is the inverse of (A'), directly from the elements b_{ij} of a matrix (B) which is the inverse of (A), when (A') differs from (A) only in the elements of one column, say the Sth column. The equations which form the basis of the computation are: $b'_{Sj} = b_{Sj} / \sum_{i=1}^{n} b_{Sr} a'_{rS}$, $j=1, 2, \cdots n \cdot b'_{ij} = b_{ij} - b'_{Sj} \sum_{r=1}^{n} b_{ir} a'_{rS}$, $i=1, 2, \cdots, S-1, S+1, \cdots, n$; $j=1, 2, \cdots, n$. Analogous equations are derived for the case that A and A' differ in the elements of a given row rather than a column. (Received July 5, 1949.)

587. D. M. Studley: *Structure of statistical elements.*

Research in logical semantics and in practical elementation has set forth the proposition that all words and ideas have set form. As a consequence of this universal proposition all notions and conceptions in statistics should be accessible to set-theoretic analysis and interpretation. This paper explains the results of a preliminary analysis performed on statistical notions and conceptions with a view to a proper organization of definitions and conceptions which will, it is hoped, make possible a better and simpler construction of statistics from a system of basic notions. (Received July 14, 1949.)

588*t*. J. E. Walsh: *Large sample tests and confidence intervals for mortality rates.*

In computing mortality rates from insurance data, the unit of measurement used is frequently based on number of policies or amount of insurance rather than on lives. Then the death of one person may result in several units of "death" with respect to the investigation; moreover, the number of units per individual may vary noticeably. Thus the usual large sample methods of obtaining significance tests and confidence intervals for the true value of the mortality rate are not applicable to these situations. If the number of units associated with each person in the investigation were known, accurate large sample results could be obtained; however, determination of the number of units associated with each individual would require an extremely large amount of work. This article presents some valid large sample tests and confidence intervals for the mortality rate which do not require much work and are reasonably efficient. The procedure followed consists in first dividing the risks into twenty-six subgroups on the basis of the first letter of the last name of the person insured. Some of the groups are then combined until 10 to 15 subgroups yielding approximately the same number of units are obtained. The fraction consisting of the total number of units paid divided by the total number of units exposed is computed for each subgroup. Asymptotically the resulting observations represent independent observations from continuous symmetrical populations with common median equal to the true value of the rate of mortality. Tests and confidence intervals for the rate of mortality are obtained by applying the results of the paper *Some significance tests for the median which are valid under very general conditions* (Annals of Mathematical Statistics vol. 20 (1949) pp. 64–81) to these observations. (Received July 2, 1949.)

TOPOLOGY

589. R. D. Anderson: *A nonhomeomorphic monotone interior mapping of the plane onto itself.*

The author shows that there exists a monotone interior transformation of the plane onto itself which is not a homeomorphism by construction of a continuous collection G of mutually exclusive compact nondegenerate continua filling up the plane such that with respect to its elements as points G is topologically equivalent to the plane. (Received July 18, 1949.)

590*t*. R. H. Bing: *A normal space which is not collectionwise normal.*

A space is collectionwise normal if for each collection K of mutually exclusive closed sets such that each subcollection of K has a closed sum there is a collection W of mutually exclusive open sets such that each element of K is covered by an element of W and no element of W covers two elements of K. An example is given of a normal Hausdorff space which is not collectionwise normal. (Received July 18, 1949.)

591. R. P. Dilworth: *The completion of the lattice of continuous functions.*

Let S be a topological space and let $C(S)$ denote the set of all bounded, real, continuous functions on S. In previous papers (Bull. Amer. Math. Soc. Abstracts 54-11-535 and 55-3-207) it has been shown that the normal completion of $C(S)$, as a lattice, is isomorphic to the lattice of all bounded continuous functions on a suitable compact Hausdorff space T_S. It is proved here that if S is completely regular, then T_S is the Boolean space associated with the complete Boolean algebra of regular open sets of S. (Received July 11, 1949.)

592. A. M. Gleason: *On the existence of arcs in locally compact connected groups.*

Theorem. Every locally compact connected group with more than one element contains an arc. The proof rests on the following lemma. Let G be a locally compact connected group with more than one element which has no compact connected subgroups except e. Then G contains an increasing family F_t of compact sets with non-negative real indices such that (i) $F_t \neq e$ for $t>0$, (ii) $F_0 = \bigcap_{t>0} F_t = \{e\}$, and (iii) $F_t F_u = F_{t+u}$. Since a compact connected group with more than one element contains not only an arc but even a one-parameter subgroup, the theorem is clearly true for groups with a nontrivial compact connected subgroup. On the other hand it is easy to construct an arc having given the family F_t described in the lemma. The lemma is proved by considering the family of all compact subsets of G as a semigroup. A locally compact topology compatible with the semigroup operation can be introduced and with its aid the construction of the one-parameter subsemigroup F_t can be effected. (Received July 14, 1949.)

593*t*. W. H. Gottschalk: *Asymptotic relations in topological groups.*

Let G be an additive, abelian, locally compact group, let μ be Haar measure in G, let E be a totally bounded measurable subset of G such that some translate of int E contains 0 and generates G, and let $x \in G$. That $\lim_{n\to\infty} \mu((nE) \cap (nE+x))/\mu(nE) = \lim_{n\to\infty} \mu((n+1)E)/\mu(nE) = 1$ is proved. (Received July 14, 1949.)

594*t*. Meyer Jerison: *The space of bounded maps into a Banach space.*

Let B^X be the Banach space of continuous maps from a compact space X into a real Banach space B. It is not true, in general, that (*) if B^X is equivalent to B^Y, then X is homeomorphic to Y. By generalizing Eilenberg's proof for real-valued functions (Ann. of Math. vol. 43 (1942) pp. 568–579), (*) is proved for a space B with the property that if two maximal convex subsets of the surface of the unit sphere intersect, there is a third one that meets neither of them. A characterization of B^X with strictly convex B is given, using methods of Myers (Ann. of Math. vol. 49 (1948)) and of Arens and Kelley (Trans. Amer. Math. Soc. vol. 62 (1947)). If B is strictly convex and T is an equivalence of B^Y onto B^X, then $(T\beta)(x) = \omega(x)[\beta(h(x))]$

for all $\beta \in B^Y$, $x \in X$, where h is a homeomorphic mapping of X onto Y and ω is a continuous mapping of X into the space of equivalences of B onto itself. (Received July 15, 1949.)

595t. W. S. Massey: *Homotopy groups of triads*. II. Preliminary report.

Let $X^* = X \cup E^n$ be a space obtained by adjoining an n-cell E^n to a topological space X, and let $\dot{E}^n = X \cap E^n$ denote the boundary of the cell E^n. In a recent paper (*The homotopy groups of a triad*, Proc. Nat. Acad. Sci. U.S.A. vol. 35 (1949) pp. 322–328) the following theorem was stated: If the pair $(X, \dot{E}^n)$ is m-connected, then the triad $(X^*; X, E^n)$ is $(m+n-1)$-connected. In this paper the problem of determining the first non-vanishing homotopy group of the triad $(X^*; X, E^n)$ is solved under fairly general conditions as follows. Assume that: (1) X^* is a topological space which can be subdivided into a finite simplicial complex in such a way that X and E^n are subcomplexes. (2) The boundary E^n of the cell E^n is an $(n-1)$-sphere. (3) The pair $(X, \dot{E}^n)$ is m-connected $(m>0)$. Then the first non-vanishing homotopy group of the triad $(X^*; E^n, X)$, that is, $\pi_{m+n}(X^*; E^n, X)$, is isomorphic to the group $\pi_{m+1}(X, \dot{E}^n)$. The functional cup product, recently introduced by Steenrod, is used in setting up the isomorphism. This theorem includes as a special case the Freudenthal "Einhängung" theorems in the critical dimensions. (Received July 14, 1949.)

596. S. B. Myers: *Functional uniformities*.

Let G be a family of sets of real continuous functions on a completely regular space X, such that each $g \in G$ is equicontinuous and G separates X (in the sense that for each $P \in X$ and open set $N(P)$ there is a $g \in G$ and $\delta > 0$ such that if $|f(Q)-f(P)| < \delta$ for all $f \in g$, then $Q \in N(P)$). Such a family G determines a uniform structure ("functional uniformity") V compatible with the topology of X as follows: for each $g \in G$ and $\delta > 0$ let $V_\alpha = \{P, Q \in X \mid |f(Q)-f(P)| < \delta \text{ for all } f \in g\}$. A functional uniformity is *bounded* if all members of every $g \in G$ are bounded, *weak* if every g is a finite set. Then it is shown that every uniform structure on X is isomorphic to a bounded functional uniformity; every precompact uniform structure is isomorphic to a weak bounded functional uniformity, and conversely every weak bounded functional uniformity is precompact. Weak functional uniformities are investigated, as well as the case that each $g \in G$ is countable. (Received July 18, 1949.)

597. O. M. Nikodým: *On linear functionals*.

Relationships between pseudotopologies and integral representations of linear functionals are studied. (Received July 12, 1949.)

598t. O. M. Nikodým: *On "pseudotopology" and integration related to it*.

A modification, termed "pseudotopology," of the ordinary open-set topology for abstract sets is axiomatized and properties of corresponding pseudo-continuous functions examined. Relationships between pseudotopologies and more general families of functions are considered. A general notion of integral and summability related to a pseudotopology is defined and examined. (Received July 12, 1949.)

599. L. T. Ratner: *Semi-continuous multi-valued transformations*.

This paper considers upper and lower semi-continuous multi-valued transformations of one weakly separable Hausdorff space into another such space. Basic properties of semi-continuous multi-valued transformations are explored and extended. Sufficient conditions for preserving connectedness and compactness are given. Results are established giving information as to the character of limits of various types of sequencies of multi-valued transformations. Finally, consideration of semi-continuity in metric spaces leads to analogues of a number of the standard theorems of real variables and of topology. (Received July 18, 1949.)

600. C. N. Reynolds: *An analysis of recent progress on the problem of coloring maps in four colors.*

This is a brief analysis, particularly of the work of S. M. de Backer, implying that those maps which are irreducible with respect to the four-color problem have at least 84 regions. (Received August 31, 1949.)

601*t*. W. R. Utz: *Unstable homeomorphisms.*

If X is a metric (d) space, the homeomorphism $f(X)=X$ is called unstable provided there exists a number $\delta(f, X)>0$ such that corresponding to each pair of distinct points x_1, x_2 of X, there is an integer $n(x_1, x_2)$ for which $d(f^n(x_1), f^n(x_2))>\delta$; the orbits of the points x_1, $x_2 \in X$ are called positively (negatively) asymptotic if for each $\epsilon>0$ there exists an integer N such that $i>N(i<N)$ implies $d(f^i(x_1), f^i(x_2))<\epsilon$. It is shown that if X is compact, dense-in-itself, and metric, and $f(X)=X$ is unstable, then some pair of distinct points of X have orbits asymptotic in at least one sense. Some properties of unstable homeomorphisms are established and examples are cited. (Received July 18, 1949.)

602. P. A. White: *On the union of two generalized manifolds.*

A closed subset M of the compact Hausdorff space S whose boundary relative to S is F is a generalized n-manifold with boundary relative to S if (a) dim $M=n$, (b) M is i-colocally-connected $(0\leqq i\leqq n-1)$, (c) M is i-locally connected $(0\leqq i\leqq n)$, (d) for each point of M the local n-betti number relative to F is 1, (e) for each point in F the local n-betti number is 0. (The local n-betti number of $p\in A\subset S$ relative to $B\subset A$ is the smallest positive integer k such that to any open set P of S containing p, there corresponds an open set Q with $p\in Q\subset P$ such that any $(k+1)$ n-dimensional Čech cycles on A mod $(S-P)\cup B$ are linearly independent on A mod $(S-Q)\cup B$.) An n-manifold M is closed if $F=0$, and is orientable if the n-betti number of M mod F is 1 irreducibly. It is shown that if M_1 and M_2 are generalized n-manifolds with a common boundary M_{12} relative to S such that $M_1\cap M_2=M_{12}$, and if M_{12} is a generalized $(n-1)$-manifold with boundary relative to $(\overline{S-M})\cup M_{12}$, then $M=M_1\cup M_2$ is a generalized n-manifold with boundary relative to S. Furthermore if M_1, M_2, and M_{12} are orientable, so is M; if M_{12} is a closed manifold, so is M. (Received May 20, 1949.)

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