PARTITIONING A SET

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1. Introduction. Suppose space has a metric $D(x, y)$.

Under certain conditions in calculus we define $\int_b^a f(x)dx$ to be equal to $\lim \sum f(\xi_i)\Delta x_i$. The dividing of the interval from a to b into a finite number of pieces with lengths Δx_i is a partitioning of the interval from a to b. Similarly, partitioning may be used in integrating over an arbitrary set. The partitioning may provide a basis for a measure on the range of the function which is being integrated. We shall define partitioning on sets for which there is no accepted unit of measure such as length, area, or volume.

The notion of partitioning may be used to assign a convex metric to certain sets. See Theorems 7, 8, 9, and 10. Theorem 8 provides an answer to a question raised by Menger in [3].[1] See [1] for a discussion of and references to partial solutions of this problem. Further applications of partitioning are treated in my paper *Complements of continuous curves*, appearing in Fund. Math.

Most of the results of this paper (including Theorem 7 but not Theorems 1 and 8) were contained in an address I gave before the Mathematics Club at the University of Chicago in May, 1948. After finishing Theorems 1 and 8 in August, I found that Moise, working independently, had obtained a solution [4] to the above mentioned problem by Menger six weeks earlier.

Partitioning. A set M can be partitioned if for each positive number ϵ there is a finite collection G of connected mutually exclusive open subsets of M such that each element of G is of diameter less than ϵ and the sum of the elements of G is dense in M. We shall say that G is an ϵ-partitioning of M.

Refinement. If G and H are two partitionings of M, we say that G is a refinement of H if each element of G is a subset of an element of H.

Property S. A set M has property S if for each positive number ϵ, M is the sum of a finite number of connected subsets such that the diameter of each of these subsets is less than ϵ. This property is discussed in [5, 6, 7, 8].

Convex metric. A metric E for a set M is convex if for each pair of points x, y of M there is a point z of M such that $E(x, z) = E(y, z)$

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

$=E(x, y)/2$. The metric F for M is almost convex [2] if for each pair of points x, y of M and each positive number ϵ there is a point z of M such that

$$|F(x, z) - F(x, y)/2| + |F(y, z) - F(x, y)/2| < \epsilon.$$

2. **Partitionable sets.** Theorem 1 characterizes sets that can be partitioned. First we consider two lemmas that will be used.

LEMMA 1. *For each connected set M with property S there is a compact locally connected continuum H and a homeomorphism T of M into a dense subset H' of H such that the diameter of each connected subset X of M is the same as the diameter of $T(X)$ and for each connected open subset R of H, $T^{-1}(R\cdot H')$ is connected.*

PROOF. The continuum H is the complete enclosure of the relative distance space for M. See [8; pp. 154–158].

LEMMA 2. *If M is a connected set with property* S *and H and K are subsets of M at a positive distance from each other, then there is a finite collection of connected, mutually exclusive open subsets of M such that the sum of this collection is dense in M, no element of this collection intersects both H and K, and each element of this collection that intersects K has property* S.

PROOF. We shall prove the lemma for the case where M is closed and compact. It will follow from Lemma 1 for the general case.

Let H' and K' be two sets at a positive distance from each other, containing H and K respectively, such that each point of M belongs to an arc of diameter less than $1/2$ that intersects $H'+K'$. Denote by H_1 (or K_1) the set of all points of M that belong to a connected subset of M which intersects H' (or K') and is of diameter less than $D(H', K')/3$. We note that H_1 and K_1 have only a finite number of components.

Let W be a finite collection of points of M such that each point of M belongs to an arc in M of diameter less than $1/4$ that intersects W. There is a finite collection A of arcs such that each element of A lies in $M-\overline{K}_1$ and intersects H_1 while A^*, the sum of the elements of A, contains each point of W which belongs to a component of $M-\overline{K}_1$ which intersects H_1. Let B be a finite collection of arcs such that A^*+B^* contains W and each element of B is of diameter less than $1/2$, lies in $M-(\overline{H}_1+A^*)$, and intersects K_1.

Denote by H_2 (or K_2) the set of all points that belong to a connected subset of M which intersects H_1+A^* (or K_1+B^*) and is of diameter less than $D(H_1+A^*, K_1+B^*)/3$. We note that each point of K_2 belongs to a connected subset of K_2 which intersects K_1 and is of diameter less than $1/2+1/6$.

Similarly, there exist open subsets $H_3, K_3, H_4, K_4, \cdots$ of M such that (1) H_i is at a positive distance from K_i, (2) H_{i+1} and K_{i+1} contain $\overline{H}_i$ and $\overline{K}_i$ respectively, (3) each component of H_{i+1} contains a component of H_i, (4) each point of K_{i+1} belongs to a connected subset of K_{i+1} that intersects K_i and is of diameter less than $1/2^i+1/(3\cdot 2^i)$, and (5) each point of M belongs to an arc of diameter less than $1/2^i$ that intersects H_i+K_i. Now $\sum(H_i+K_i)$ is dense in M and has only a finite number of components while $\sum K_i$ has property S.

THEOREM 1. *A necessary and sufficient condition that a set M can be partitioned is that it have property* S.

PROOF OF SUFFICIENCY. We show that M can be partitioned by showing that each of its components C can be partitioned.

Let $(p_1, p_2, \cdots, p_n)$ be a finite collection of points of C such that each point of M is at a distance from $\sum p_i$ of less than $\epsilon/4$. Let H_i be the set of all points of C that are at a distance from p_i of less than $\epsilon/4$ and K_i be the set of those that are at a distance from p_i of more than $\epsilon/2$.

By Lemma 2 there are two mutually exclusive open subsets U_1 and V_1 of C such that U_1+V_1 is dense in C, U_1 has a finite number of components and contains H_1, and V_1 has property S and contains K_1. Also, there are two open subsets U_2 and V_2 of V_1 such that U_2+V_2 is dense in V_1, U_2 contains $V_1\cdot H_2$ and has a finite number of components, while V_2 contains $V_1\cdot K_2$ and has property S. Similarly, we define sets $U_3, V_3, U_4, V_4, \cdots, U_{n-1}, V_{n-1}$. The components of $U_1+U_2+\cdots+U_{n-1}+V_{n-1}$ are finite in number, each is of diameter less than ϵ, and their sum is dense in M.

3. **Refinements of partitionings.** We shall be interested in partitioning a set in such a way that each of the pieces into which it is partitioned can be partitioned. These pieces must have property S if this is to be done.

THEOREM 2. *If H and K are two connected subsets of the connected partitionable set M and ϵ is a positive number such that any connected subset of M intersecting both H and K is of diameter more than ϵ, then there are two connected, mutually exclusive open subsets U and V of M containing H and K respectively such that $U+V$ has property* S *and is dense in M.*

PROOF. The general case will follow from Lemma 1 if we prove the case where M is a compact continuum. Hence, we suppose it is one.

For each integer i, let G_i be a $1/i$-partitioning of M. Let H' and K' be mutually exclusive subcontinua of M containing H and K respectively such that each element of G_2 intersects $H'+K'$. Let U_1 (or V_1) be the set of all points that belong to a connected subset of M of diameter less than $D(H', K')/3$ that intersects H' (or K').

Let A and B be finite collections of arcs such that (1) each element of A intersects U_1 while each element of B intersects V_1, (2) each element of $A+B$ is a subset of an element of G_2, (3) no element of A intersects either $\overline{V}_1$ or an element of B, while no element of B intersects $\overline{U}_1$, and (4) each element of G_4 contains a point of A^*+B^*.

Let U_2 (or V_2) be the set of all points that belong to a connected subset of M of diameter less than $D(U_1+A^*, V_1+B^*)/3$ that intersects U_1 (or V_1). We note that each point of U_2 (or V_2) belongs to a connected subset of U_2 (or V_2) of diameter less than $1/2+D(U_1+A^*, V_1+B^*)/3<1/2+1/6$ that intersects U_1 (or V_1).

Similarly, we obtain connected open subsets U_3, V_3, U_4, V_4, $\cdots$ such that (1) U_{i+1} contains $\overline{U}_i$ while V_{i+1} contains $\overline{V}_i$, (2) U_i+V_i intersects each element of G_{2^i}, and (3) each point of U_{i+1} (or V_{i+1}) belongs to a connected subset of U_{i+1} (or V_{i+1}) which is of diameter less than $1/2^i+1/(3\cdot 2^i)$ and intersects U_i (or V_i). Then $U=\sum U_i$ and $V=\sum V_i$ are the required sets.

Each of the following three theorems follows from repeated applications of the theorem preceding it.

THEOREM 3. *If H is a finite collection of connected subsets of the connected partitionable set M and ϵ is a positive number such that no connected subset of M of diameter less than ϵ intersects two elements of H, then there is a finite collection U of connected, mutually exclusive open subsets of M such that each element of U has property* S *and contains one and only one element of H, each element of H is contained in an element of U, and the sum of the elements of U is dense in M.*

THEOREM 4. *If M is partitionable, then for each positive number ϵ there is an ϵ-partitioning G of M such that each element of G has property* S.

THEOREM 5. *If M is a partitionable set, there is a sequence G_1, G_2, $\cdots$ such that G_i is a $1/i$-partitioning of M and G_{i+1} is a refinement of G_i.*

THEOREM 6. *Suppose space has a complete metric and R is a connected domain with property* S *and a nondegenerate boundary B. Then R contains a connected set E with property* S *such that $\overline{E}=E+B$, but no connected subset of E has this property. Also, E is topologically* *equivalent to a subset of a dendron, and any nondegenerate continuum in E is a dendron with a finite number of ends.*

PROOF. Let $G_1, G_2, \cdots$ be a sequence such that G_i is a $1/i$-partitioning of R and G_{i+1} is a refinement of G_i. Denote the collection of closures of elements of G_i by H_i. Let H_1' be a subset of H_1 irreducible with respect to covering B and H_{i+1}' be a subcollection of H_{i+1} such that each element of H_{i+1}' is a subset of an element of H_i', and H_{i+1}' is irreducible with respect to covering B.

Let m_1 be an integer so large that H_{m_1}' has more than one element. There is a dendron T_1 in R which intersects each element of H_{m_1}' but which contains no subcontinuum doing so. Let m_2 be an integer so large that $D(T_1, B) > 3/m_2$. There is a dendron T_2 such that (1) T_2 contains T_1, (2) T_2 intersects each element of H_{m_2}', (3) each component of $T_2 - T_1$ is a subset of an element of G_{m_1}, and (4) no subcontinuum of T_2 has these properties. Similarly, we define $T_3, T_4, \cdots$. Then $E = \sum T_i$.

4. **Convexification of partitionable sets.** We can assign a convex metric to certain partitionable sets.

THEOREM 7. *If M is a compact partitionable continuum, it can be assigned a convex metric.*

FIRST PROOF OF THEOREM 7. We shall define a sequence $E_1(x, y)$, $E_2(x, y), \cdots$ of functions on the pairs of points of M such that $E(x, y) = \lim E_i(x, y)$ is a convex metric for M.

Let $G_1, G_2, \cdots$ be a sequence such that G_i is a $1/i$-partitioning of M and G_{i+1} is a refinement of G_i. There is a dendron T_1 in M such that T_1 intersects each element of G_1, and there is a convex metric F_1 for T_1 such that the diameter of T_1 under F_1 is less than $1/4$. Let U_1 be the set of all elements u such that u is a component of $g - g \cdot T_1$ for some element g of G_1. Denote by V_1 the set of all continua v such that v is either a point of T_1 or the closure of an element of U_1.

If v and w are two elements of V_1, we define $E_1(v, w)$ to be the greatest lower bound of all numbers of the sort $r/2 + s$, where some continuum is the sum of v, w, r other elements of V_1 and some arcs in T_1 whose total length under F_1 is s.

Let m_2 be an integer so large that $D(v, w) > 3/m_2$ if v and w are elements of V_1 such that $E_1(v, w) > 1/16$. In each element u_i of U_1 that contains an element of G_{m_2} there is a dendron $T_{2,i}$ which intersects each element of G_{m_2} that intersects u_i. It may be noted that there are only a finite number of such u_i's. Let $F_{2,i}$ be a convex metric for $T_{2,i}$ such that the diameter of $T_{2,i}$ under $F_{2,i}$ is less than $1/16$.

Let U_2 be the collection of all components of the sort $g-g\cdot(T_1+\sum T_{2,i})$, where g denotes an element of G_{m_2}. Let V_2 designate the set of all continua v such that v is either a point of $T_1+\sum T_{2,i}$ or the closure of an element of U_2.

If v and w are two elements of V_2, we define $E_2(v, w)$ to be the greatest lower bound of all numbers of the type $r/4+s$, where some continuum is the sum of v, w, r other elements of V_2 and a collection of arcs in $T_1+\sum T_{2,i}$ whose total length under their respective F's is s.

Similarly, we define m_3, G_{m_3}, $T_{3,j}$, $F_{3,j}$, U_3, V_3, $E_3(v, w)$, m_4, G_{m_4}, $T_{4,j}$, $F_{4,j}$, U_4, V_4, $E_4(v, w)$, $\cdots$. In general, m_{i+1} is an integer so large that

$$(1) \qquad D(v, w) > 3/m_{i+1} \quad \text{if} \quad E_i(v, w) > 1/4^{i+1} \qquad (v,\ w \text{ elements of } V_i).$$

Consider an element $u_{i,n}$ of U_i that contains an element of $G_{m_{i+1}}$. Let $K_{i,n}$ denote the collection of all continua k such that k is the closure of a component of the common part of $u_{i,n}$ and an element of $G_{m_{i+1}}$. Denote by $K'_{i,n}$ the finite collection of all elements k of $K_{i,n}$ such that k intersects an element of V_i which does not intersect any component of $T_1+\sum T_{2,j}+\cdots+\sum T_{i,j}$ which intersects k. There is a dendron $T_{i+1,n}$ in $u_{i,n}$ such that $T_{i+1,n}$ intersects each element of $K'_{i,n}$ and each element of $G_{m_{i+1}}$ which intersects $u_{i,n}$. Let $F_{i+1,n}$ be a convex metric for $T_{i+1,n}$ such that

$$(2) \qquad \text{diameter } T_{i+1,n} \text{ under } F_{i+1,n} < 1/4^{i+1}.$$

Denote by U_{i+1} the collection of all components of sets of the sort $g-g\cdot(T_1+\sum T_{2,j}+\cdots+\sum T_{i+1,j})$, where g denotes an arbitrary element of $G_{m_{i+1}}$. The elements of V_{i+1} are the points of $T_1+\sum T_{2,j}+\cdots+\sum T_{i+1,j}$ and the closures of the elements of U_{i+1}.

Let

$$E_{i+1}(v, w) = \text{greatest lower bound } (r/2^{i+1}+s) \quad (v,\ w \text{ elements of } V_{i+1}),$$

where some continuum is the sum of v, w, r other elements of V_{i+1} and a collection of arcs in $T_1+\sum T_{2,j}+\cdots+\sum T_{i+1,j}$ whose lengths under their respective F's total s. It may be noted that if v intersects w, $E_i(v, w)=0$.

If p and q are two points of M, we define $E_i(p, q)$ to be the greatest lower bound of all numbers $E_i(v, w)$, where v and w are elements of V_i containing p and q respectively. We shall show that $\lim E_i(p, q) = E(p, q)$ exists and $E(x, y)$ is a convex metric for M.

Using (1) we find that if some continuum is the sum of the elements v and w of V_i and n elements of V_{i+1}, then $E_i(v, w) < n/2^{i+1}$. From this and the fact that two elements of V_{i+1} are subsets of the same element of V_i if they intersect the same arc in $\sum T_{i+1,j}$, we find that if v and w are two elements of V_{i+1}, then there are elements v' and w' of V_i containing v and w respectively such that $E_i(v', w') \leqq E_{i+1}(v, w)$. Hence, $E_i(p, q)$ is a monotone nondecreasing function of i.

If a component C of $T_1 + \sum T_{2,j} + \cdots + \sum T_{i,j}$ intersects the sum of two elements v, w of V_i which intersect each other, it follows from (1) and (2) that under its respective metric

$$\text{diameter } C \cdot (v + w) < 1/4^i. \tag{3}$$

If v and w are two elements of V_{i+1} which are subsets of the same element of V_i, then

$$E_{i+1}(v, w) < 1/4^{i+1} + \delta(1/2^{i+1} + 1/4^i) \tag{4}$$

where δ is equal to 2, 1, or 0 according as neither, one, or both of the elements v, w are nondegenerate subsets of elements of $K'_{i,j}$ for some integer j.

Using (3) and (4) we find that if v and w are elements of V_{i+1} which are subsets of the sum of a coherent collection of n elements of V_i, then

$$E_{i+1}(v, w) < n(1/2^i + 1/4^i + 1/4^{i+1}) + 1/4^i < (n/2^i)(1 + 3/2^i). \tag{5}$$

Using (5) we find that

$$E_{i+1}(p, q) < (1 + 3/2^i)[E_i(p, q) + 2/2^i]. \tag{6}$$

Repeated applications of (6) give that

$$E_{i+k}(p, q) < (1 + 3/2^i)(1 + 3/2^{i+1}) \cdots (1 + 3/2^{i+k-1})[E_i(p, q) + 2/2^i + 2/2^{i+1} + \cdots + 2/2^{i+k-1}].$$

Since

$$(1 + 3/2^i)(1 + 3/2^{i+1}) \cdots < [1 + 3/(2^i - 6)][1 + 3/(2^{i+1} - 6)] \cdots = 2^i/(2^i - 6) \qquad (i > 3),$$

then

$$E_i(p, q) \leqq E(p, q) < [2^i/(2^i - 6)][E_i(p, q) + 4/2^i] \qquad (i > 3). \tag{7}$$

That $\lim E_i(p, q) = E(p, q)$ exists follows from the fact that $E_1(p, q)$, $E_2(p, q), \cdots$ is a bounded monotone sequence.

That $E(x, y)$ satisfies the triangle condition follows from the fact that for each triple of points p, q, r, $E_i(p, q) \leqq E_i(p, r) + E_i(r, q) + 1/2^i$. Since $E_i(x, y)$ is symmetric, $E(x, y)$ is also; that is, $E(x, y) = E(y, x)$. If R is an open subset of M containing the point p, there is an integer n so large that $D(p, M-R)>3/m_n$. Since $E_n(p, M-R)$ is positive, $E(p, M-R)$ is also. If p and q belong to the closure of the same element of G_{m_n+1}, then by (4), $E_n(p, q)<2/2^n$, and by (7), $E(p, q) < [2^n/(2^n-6)](2/2^n+4/2^n)$ if $n>3$. Hence $E(p, q_1)$, $E(p, q_2)$, $\cdots$ has zero as a limit if and only if $q_1, q_2, \cdots$ converges to p. Therefore $E(x, y)$ is a metric for M that preserves its topology.

That $E(x, y)$ is almost convex follows from the fact that for each pair of points p, q and each integer n there is a point r such that $|E_n(p, r)-E_n(p, q)/2|+|E_n(r, q)-E_n(p, q)/2| \leq 1/2^n$. Since M is compact, $E(x, y)$ is a convex metric for M.

ALTERNATE PROOF OF THEOREM 7. The proof of Theorem 7 can be simplified by making use of Theorem 6 of this paper and Theorem 4 of [1] which states that *a compact locally connected continuum M has a convex metric if for each point p contained in an open subset R_1 of M, there is an open subset R_2 of R_1 containing p such that the boundary of R_2 with respect to M is a subset of a subcontinuum of M with a convex metric.* Hence, we need to show only that $\bar{E}$ of Theorem 6 has a convex metric.

Now E is the sum of a dendron T_1 and a finite collection U_1 of connected, mutually exclusive open subsets of E such that the common part of T_1 and the closure of an element of U_1 is an end point of T_1. There is a convex metric F_1 for T_1 such that the distance between any two of its end points is between 1/2 and 1. Let V_1 be the collection of all continua v such that v is either a point of T_1 or the closure of an element of U_1. If v and w are elements of V_1, define $E_1(v, w)$ to be the greatest lower bound of all numbers of the type $r/2+s$ where some continuum is the sum of v, w, r other elements of V_1 and arcs in T_1 whose lengths under F_1 total s.

Let ϵ_2 be a number so small that if v and w are two elements of V_1 which do not intersect, then $D(v, w)>3\epsilon_2$ and $D(T_1, \bar{E}-E)>3\epsilon_2$. Now E is the sum of a dendron T_2 and a finite collection U_2 of connected, mutually exclusive open subsets of E such that each element of U_2 is of diameter less than ϵ_2, and the common part of T_2 and the closure of an element of U_2 is an end point of T_2. Let F_2 be a convex metric for T_2 that preserves F_1 on T_1 and such that any arc in T_2 is of length between $1/2^2$ and 1/2 if the end points of this arc are also end points of the closure of a component of T_2-T_1. Denote by V_2 the collection of all elements v such that v is either a point of T_2 or the closure of an element of U_2. If v and w are elements of V_2, let $E_2(v, w)$ be the greatest lower bound of numbers of the type $r/4+s$, where some continuum is the sum of v, w, r other elements of V_2 and arcs in T_2 whose lengths under F_2 total s.

This process is continued to get a sequence of functions $E_3, E_4, \cdots$. In general, ϵ_{i+1} is a number so small that if v and w are any two elements of V_i that do not intersect each other, then $D(v, w) > 3\epsilon_{i+1}$ and $D(T_i, \overline{E}-E) > 3\epsilon_{i+1}$. There is a dendron T_{i+1} and a finite collection U_{i+1} of connected, mutually exclusive open subsets of E such that each element of U_{i+1} is of diameter less than ϵ_{i+1}, and the common part of T_{i+1} and the closure of an element of U_{i+1} is an end point of T_{i+1}. There is a convex metric F_{i+1} for T_{i+1} that preserves F_i on T_i such that any arc in T_{i+1} is of length, under F_{i+1}, of between $1/2^{i+1}$ and $1/2^i$ if the end points of this arc are end points of the closure of the same component of $T_{i+1}-T_i$. Use V_{i+1} to denote the collection of points of T_{i+1} and the closures of elements of U_{i+1}. For elements v and w of V_{i+1}, define $E_{i+1}(v, w)$ to be the greatest lower bound of $r/2^{i+1}+s$, where some continuum is the sum of v, w, r other elements of V_{i+1} and arcs in T_{i+1} whose lengths under F_{i+1} total s.

Define $E_i(p, q)$ to be the greatest lower bound of $E_i(v, w)$ where p and q are points of elements v and w respectively of V_i. The argument to show that $\lim E_i(p, q)$ exists and determines a convex metric for $\overline{E}$ is similar to and somewhat easier than that used in the first proof.

Applying Theorem 1 to Theorem 7 we obtain the following.

THEOREM 8. *Each compact, locally connected continuum has a convex metric.*

The application of Lemma 1 and Theorem 1 to Theorem 7 gives Theorems 9 and 10.

THEOREM 9. *Each connected partitionable set has an almost convex metric.*

THEOREM 10. *Each connected set with property* S *has an almost convex metric.*

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