ON INTERPOLATION TO AN ANALYTIC FUNCTION IN EQUIDISTANT POINTS: PROBLEM β

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The present note is an addendum to a paper by the same authors,[1] to which (especially §8) the reader may refer for detailed notation and definitions.

Two phases of the direct problem of the study of degree of convergence of a sequence of functions approximating to a given function are (i) proof of the existence of a sequence approximating with a certain degree of convergence and (ii) study of the degree of convergence of a sequence defined by a specific method. We are here concerned with the second phase of the problem:

THEOREM 1. *Let the function $f(z)$ be analytic and of class $L(p, \alpha)$ in the annulus $\gamma_\rho: \rho > |z| > 1/\rho < 1$, where ρ is given. Let*

$$p_n(z) = a_{nn}z^n + a_{n,n-1}z^{n-1} + \cdots + a_{n0} + \cdots + a_{n,-n}z^{-n}$$

be the unique polynomial in z and $1/z$ of degree n which coincides with $f(z)$ in the $(2n+1)$st roots of unity. Then for z on $\gamma: |z| = 1$ we have

$$(1) \qquad |f(z) - p_n(z)| \leqq M/\rho^n \cdot n^{p+\alpha},$$

where M is independent of n and z.

The polynomial $p_n(z)$ may be defined by the equation

$$(2) \qquad f(z) - p_n(z) = \frac{1}{2\pi i}\int_{|z|=r} + \int_{|z|=1/r} \frac{t^n(z^{2n+1} - 1)f(t)dt}{z^n(t^{2n+1} - 1)(t - z)},$$

for z in the annulus $1 > r > |z| > 1/r$; the function $f(z)$ is assumed analytic for $r > |z| > 1/r$, continuous for $r \geqq |z| \geqq 1/r$. In particular, if $f(z)$ itself is here chosen as a polynomial $P_n(z)$ in z and $1/z$ of degree n, the interpolating polynomial $p_n(z)$ must coincide with $P_n(z)$, so we have for $r > |z| > 1/r$

$$(3) \qquad 0 = \frac{1}{2\pi i}\int_{|t|=r} + \int_{|t|=1/r} \frac{t^n(z^{2n+1} - 1)P_n(t)dt}{z^n(t^{2n+1} - 1)(t - z)}.$$

Return to the original function $f(z)$ in (2) with subtraction of (3) from (2) thus yields for $r > |z| > 1/r$

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[1] Trans. Amer. Math. Soc. vol. 49 (1941) pp. 229–257.

$$(4)\quad f(z) - p_n(z) = \frac{1}{2\pi i}\int_{|z|=r} + \int_{|z|=1/r} \frac{t^n(z^{2n+1} - 1)[f(t) - P_n(t)]dt}{z^n(t^{2n+1} - 1)(t - z)}.$$

Suppose now $f(z)$ is of class $L(p, \alpha)$ in the annulus $\rho > |z| > 1/\rho < 1$, where p is non-negative, $0 < \alpha \leqq 1$. Then by Cauchy's integral formula for the annulus, $f(z)$ can be written $f(z) = f_1(z) + f_2(z)$, where $f_1(z)$ is analytic in the region $|z| < \rho$ and is of class $L(p, \alpha)$ on $|z| = \rho$, and where $f_2(z)$ is analytic in the region $|z| > 1/\rho$ even at infinity, and $f_2(1/z)$ is of class $L(p, \alpha)$ on $|z| = \rho$. Consequently (the result is due to Curtiss) there exist polynomials $P_n'(z)$ and $P_n''(z)$ in z and $1/z$ respectively of degree n such that we have

$$|f_1(z) - P_n'(z)| \leqq M_1/n^{p+\alpha}, \qquad \text{for } |z| = \rho,$$

$$|f_2(z) - P_n''(z)| \leqq M_2/n^{p+\alpha}, \qquad \text{for } |z| = 1/\rho.$$

By the principle of maximum modulus we may write

$$|f_1(z) - P_n'(z)| \leqq M_1/n^{p+\alpha}, \qquad \text{for } |z| = 1/\rho,$$

$$|f_2(z) - P_n''(z)| \leqq M_2/n^{p+\alpha}, \qquad \text{for } |z| = \rho.$$

Then if we set $P_n(z) = P_n'(z) + P_n''(z)$, the function $P_n(z)$ is a polynomial in z and $1/z$ of degree n, and on both $|z| = \rho$ and $|z| = 1/\rho$ we have

$$(5)\qquad |f(z) - P_n(z)| \leqq (M_1 + M_2)/n^{p+\alpha}.$$

When $|t| = \rho > 1$, we have $|t^n/(t^{2n+1} - 1)| < m/\rho^n$, where m is independent of t and n, and we have a similar relation for $|t| = 1/\rho$, so inequality (5) and equation (4) with $r = \rho$ yield the conclusion of the theorem, $p \geqq 0$.

Suppose now that $f(z)$ is of class $L(p, \alpha)$ in the annulus $\rho > |z| > 1/\rho$, where p is negative; we have by definition $(r < \rho)$

$$|f(z)| \leqq N \cdot (\rho - r)^{p+\alpha}$$

for $|z| = r > 1$ and for $|z| = 1/r$, where N is independent of z and r. If we set $r = r_n = (1 - 1/n)\rho$ directly in (2), we also obtain the conclusion of the theorem; compare §3 of the paper already mentioned. The proof is now complete.

The degree of approximation asserted in Theorem 1 is also the degree of approximation obtained (loc. cit.) for the partial sums of the Laurent series for $f(z)$.

If $f(z)$ is an arbitrary function continuous on γ, and if for the polynomials $t_n(z)$ in z and $1/z$ of degree n of best approximation to $f(z)$ on γ in the sense of Tchebycheff we have

$$|f(z) - t_n(z)| \leqq \epsilon_n, \qquad z \text{ on } \gamma,$$

then for the polynomials $p_n(z)$ in z and $1/z$ of degree n which interpolate to $f(z)$ in the $(2n+1)$st roots of unity, we have[2]

$$|f(z) - p_n(z)| \leqq \epsilon_n \log n, \qquad z \text{ on } \gamma.$$

Theorem 1 represents an improvement on this result for the special functions $f(z)$ of class $L(p, \alpha)$ in $\rho \geqq |z| \geqq 1/\rho$, provided $0<\alpha<1$, because for such functions we have in the above notation

$$|f(z) - t_n(z)| \leqq M/\rho^n \cdot n^{p+\alpha}, \qquad z \text{ on } \gamma,$$

an inequality which for the case $0<\alpha<1$ and for the entire class of functions cannot be improved.

We formulate explicitly two consequences of our preceding discussion:

COROLLARY 1. *Let $f(z)$ be analytic in the annulus $\rho>|z|>1/\rho<1$' continuous in the corresponding closed region, and let us suppose polynomials $P_n(z)$ in z and $1/z$ of degree n to exist such that we have*

$$(6) \qquad |f(z) - P_n(z)| \leqq \epsilon_n, \qquad \textit{for } |z| = \rho \textit{ and } |z| = 1/\rho.$$

Then for the polynomials $p_n(z)$ defined in Theorem 1 we have

$$|f(z) - p_n(z)| \leqq M\epsilon_n/\rho^n, \qquad \textit{for } |z| = 1,$$

where M is independent of n and z.

A sufficient condition for (6) is that $f(z)$ can be written $f(z) \equiv f_1(z) + f_2(z)$ in the closed region $\rho \geqq |z| \geqq 1/\rho$, where $f_1(z)$ is analytic for $|z| <\rho$, and $f_2(z)$ is analytic for $|z| >1/\rho$ including the point at infinity, and that polynomials $P_n'(z)$ and $P_n''(z)$ in z and $1/z$ respectively of degree n exist so that we have

$$|f_1(z) - P_n'(z)| \leqq \epsilon_n, \qquad \text{for } |z| = \rho,$$
$$|f_2(z) - P_n''(z)| \leqq \epsilon_n, \qquad \text{for } |z| = 1/\rho.$$

COROLLARY 2. *Let $f(z)$ be analytic in the annulus $\rho>|z|>1/\rho<1$, let the function $\phi(r)$ be defined in the interval $1<r<\rho$, and suppose we have $|f(z)| \leqq \phi(\rho-r)$ on the two circles $|z| = r$ and $z = 1/r$. Then for the polynomials $p_n(z)$ defined in Theorem 1 we have*

$$|f(z) - p_n(z)| \leqq M \cdot \phi(\rho/n)/\rho^n, \qquad \textit{for } |z| = 1,$$

[2] D. Jackson, *The theory of approximation*, Amer. Math. Soc. Colloquium Publications, vol. 11, p. 121.

where M is independent of n and z.

There can be developed extensions of the above results to approximation on an arbitrary analytic Jordan curve or on more general point sets bounded by analytic Jordan curves, by rational functions with poles in prescribed points or uniformly distributed on given curves. The present results are intended primarily as illustrations of this general theory.

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NOTE ON THE COEFFICIENTS OF THE CYCLOTOMIC POLYNOMIAL

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Erdös[1] has proved that if A_n denotes the largest coefficient (in absolute value) of the nth cyclotomic polynomial, then for infinitely many n

$$A_n > \exp\left\{c_1(\log n)^{4/3}\right\}.$$

He also conjectured that a much stronger statement may be true, namely that[2]

$$\text{(A)} \qquad A_n > \exp\left\{n^{(c_{13}/\log\log n)}\right\}$$

holds for some c_{13} and infinitely many n, but pointed out that this would be a best result, since

$$\text{(B)} \qquad A_n < \exp\left\{n^{(c_{14}/\log\log n)}\right\}$$

for some c_{14} and all n. Erdös suppressed the proof of (B), because his proof was complicated. It is the purpose of this note to give the following short proof of (B).

The cyclotomic polynomial $F_n(x) = \prod_{d|n}(1-x^d)^{\mu(n/d)}$ is majorized by the power series

$$\prod_{d|n}(1 + x^d + x^{2d} + \cdots).$$

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[1] Paul Erdös, *On the coefficients of the cyclotomic polynomial*, Bull. Amer. Math. Soc. vol. 52 (1946) pp. 179–184.

[2] Formulas (A) and (B) were printed incorrectly in Erdös' paper (on the bottom of p. 182).