chapter of the book, and is full of useful information. The thoroughness in laying the groundwork in chapters VIII and IX simplifies the work of this chapter considerably.

XI. Integral representations with integrals of confluent hypergeometric functions. This chapter has been added in the second edition.

As J. H. Curtiss remarks in the Foreword, this work is "a labor of love on the part of the author" with whom "to derive new formulas pertinent to the hypergeometric function was, quite literally, his hobby as well as his profession." The resulting book has a flavor all of its own which sets it apart from all other books on the subject. The monograph is written by a specialist for mathematicians seeking highly specialized information; it does not attempt to replace, or to compete with, standard texts, and offers much that will be new even to experts in this field. The continuing demand for the book is due in a large measure to the increasing number of mathematicians who have "discovered" Snow's monograph and found it so helpful that they would like to own a copy. In the course of the last ten years the present reviewer loaned his copy of the first edition to numerous colleagues, and on the book being returned (somewhat reluctantly in many cases), the borrower almost invariably asked how he could buy a copy. It is a pleasant thought that in the future it will be possible to give a simple answer to this question.

A. ERDÉLYI

Introduction to the theory of stochastic processes depending on a continuous parameter. By H. B. Mann. (National Bureau of Standards Applied Mathematics Series, no. 24.) Washington, Government Printing Office, 1953. 45 pp. 30 cents.

To have written a serious textbook on the theory of stochastic processes in the small compass of forty-five pages is an astonishing *tour de force* and the reviewer is full of admiration for the ingenuity with which so much has been packed into so small a space. But brevity can be an enemy of clarity and even the "educated mathematician" to whom (in the foreword by Dr. J. H. Curtiss) the argument is addressed may find some difficulty in discovering from the evidence presented what a stochastic process *is* (unless he already knows). The following notes, supplementing Chapter I, may help in pinpointing the author's point of view in relation to other surveys.

Professor Mann starts by defining an (indexed) family of random variables $\{x_t: t\in T\}$, and he calls it a stochastic process if the index-set T is a set of real numbers. If we ignore this distinction (and it is not a useful one) then we can say with the author that a stochastic process has been defined when we are given:

(i) an index-set T;

(ii) for each $n \geqq 1$ and for each $(t_1, \cdots, t_n) \in T^n$ an n-dimensional distribution-function $F_{t_1,\cdots,t_n}$;

(iii) consistency relations of the two types,

$$F_{t_1,\cdots,t_n,t}(\xi_1, \cdots, \xi_n, \infty) = F_{t_1,\cdots,t_n}(\xi_1, \cdots, \xi_n)$$

and

$$F_{s_1,\cdots,s_n}(\eta_1, \cdots, \eta_n) = F_{t_1,\cdots,t_n}(\xi_1, \cdots, \xi_n)$$

where $(s_1, \cdots, s_n)$ is any permutation of $(t_1, \cdots, t_n)$ and $(\eta_1, \cdots, \eta_n)$ is the same permutation of $(\xi_1, \cdots, \xi_n)$.

Abstract objects $\{x_t: t \in T\}$ are then introduced so that (whenever A is an n-dimensional Borel set) we can write

$$P\{[x_{t_1}, \cdots, x_{t_n}] \in A\}$$

instead of

$$\int_A dF_{t_1,\cdots,t_n};$$

these objects x_t are called the "random variables" composing the family, but the customary language of probability theory is employed only as a colourful way of making statements about the distribution-functions F.

The justification for this procedure is not explained in the book but it is, of course, the theorem of Daniell and Kolmogorov. Let Ω be the set of all mappings ω of T into the real line. Then $\omega(t)$ can be regarded either as the value of the real-valued function $\omega(\cdot)$ at t or as the value of the real-valued function $\cdot(t)$ at ω. A Borel field $\mathcal{F}$ of subsets of Ω is defined by requiring it to be the smallest Borel field with regard to which every one of the functions $\cdot(t)$ is measurable. Their theorem then asserts, given (i) to (iii) above, that there exists a unique probability-measure P on $(\Omega, \mathcal{F})$ such that, for each $n \geqq 1$, for each $(t_1, \cdots, t_n) \in T^n$, and for each n-dimensional Borel set A,

$$P\{[\omega(t_1), \cdots, \omega(t_n)] \in A\} = \int_A dF_{t_1,\cdots,t_n}.$$

Thus the indexed family of abstract random variables $\{x_t: t \in T\}$ can be identified with the indexed family $\{\cdot(t): t \in T\}$ of real-valued

$\mathcal{J}$-measurable functions with domain Ω. It is usual to complete $\mathcal{J}$ under P and the resulting triple $(\Omega, \mathcal{J}, P)$ is then a fundamental probability space (f.p.s.) which could have been taken as the starting point of the discussion (and is so taken, for example, in the recent book by Doob).

The more usual procedure is in fact to start with a f.p.s. $(\Omega, \mathcal{J}, P)$ and then to say that a stochastic process is any (indexed) family of random variables (measurable functions). The insistence on indexing is interesting. It is trivial that an indexing can always be supplied (e.g., by well-ordering) and thus it is sometimes useful to speak of the "universal" stochastic process consisting of all measurable functions on the f.p.s.

But in most cases the index-set serves an end more important than a mere labelling, and is itself endowed with a significant structure. For example, T may be ordered and the process Markovian, or T may be an abelian group and the process stationary, or T may be a topological space and the process continuous in some sense or other.

If the process is to be continuous then reference must be made to one of the stochastic topologies. Of these the most important are those associated with the following modes of convergence:

(a) convergence in probability (in measure);

(b) convergence in mean square;

(c) convergence with probability one (almost everywhere).

Now the statement "$x_{t_n} \to x_t$ as $n \to \infty$" can be expressed in terms of the finite-dimensional distributions F in cases (a) and (b) only, and this is why, in the work under review, no reference is made to the third mode of convergence. Chapter I begins with a detailed study of modes (a) and (b) and a completeness theorem is proved for each. Here the author's approach, specially constructed to avoid measure-theoretic difficulties, leads him at once into difficulties of another sort; the trouble is that the limiting random variable (whose existence is to be proved in the completeness theorem) will not normally be a member of the family of random variables with which he starts. He is not allowed to say that it is some measurable function on the f.p.s. (and so a member of the "universal" stochastic process mentioned above) and the following complicated evolutions are necessary. A new abstract object τ is introduced and adjoined to T to give an enlarged index-set T^+. A new stochastic process $\{y_u : u \in T^+\}$ with index-set T^+ is then defined in such a way that its natural contraction to T is identifiable with the original process, so that any Cauchy sequence $\{x_{t_n}\}$ can be identified with a Cauchy sequence $\{y_{t_n}\}$ from the new process. Finally it is shown that when the extension is suitably carried out, we shall have $y_{t_n} \to y_\tau$, and this is the form taken by the completeness theorem. The disadvantage of this procedure is that a new family of random variables has to be introduced with every limit-operation, a difficulty which could have been avoided by introducing a family of random variables closed under all the modes of stochastic convergence to be considered. This could have been done in several ways and would not seem to present any difficulties of principle, but it is the price which must be paid for the avoidance of measure theory, and the reader may find Chapter I quite mystifying until this fact is realized.

The last five words in the title to the work mean that attention is to be focused on those stochastic processes $\{x_t : t \in T\}$ for which T is a finite or infinite real interval. Thus it is natural to expect an analogue to the classical theory of functions of a real variable for the "function" $x_.$, and its continuity, differentiation, and integration with regard to t are discussed in the remainder of Chapter I. There then follow four chapters devoted to processes having independent increments, and their close relatives. Here the essential idea is that of a family of random variables with an ordered index-set T and such that the differences $x_{t_2} - x_{t_1}, \cdots, x_{t_n} - x_{t_{n-1}}$ are mutually independent whenever the intervals $(t_1, t_2), \cdots, (t_{n-1}, t_n)$ are non-overlapping. This branch of the subject has a long history starting with remarkable work by Bachelier and Wiener and developed extensively by Paul Lévy and others. The general theory is presented here in Chapter IV together with an account of infinitely divisible distribution-functions. The Wiener process (in which the independent increments have Gaussian distributions) is dealt with in Chapter II and called the fundamental random process (f.r.p.); a more realistic model for the Brownian motion (associated with the names of Ornstein and Uhlenbeck and so here called the o.u.p.) is also discussed at some length. Chapter III is devoted to the problems of estimating the parameters in a f.r.p. or an o.u.p. when a single realization of the process is available, and of estimating a deterministic motion masked by the superposition of a f.r.p. In Chapter V the author discusses two ways in which a process having independent increments may be modified by a registering device. In the first of these the original process is replaced by a moving average of itself; in the second a Geiger counter transforms a process initially of the Poisson type.

The last chapter contains a brief selection of results concerning the harmonic analysis of a stochastic process, when this is of the second order (i.e., when $\int \xi^2 dF_t(\xi)$ is finite for every t). The reader of this part of the book may easily not realize how much more is known and for further information he should be referred to the longer book by Doob (which is now available but was not when Professor Mann's book was published); there he will find these problems treated in their Hilbert space versions. The particular topics covered by the work under review are (i) the expansion of x_t as a Fourier series in t (with random coefficients) over a finite interval, the Paley-Wiener expansion for the f.r.p. being worked out as an example, and (ii) the mean ergodic theorem.

To summarize: this is a very useful little book and everyone concerned with the subject will want to possess it for the light which a study of it throws on the larger works by Lévy and Doob. But for all its brevity it does not make easy reading; the approach adopted is a difficult one to follow conscientiously; and the reader should be warned that many of the best things now known are not referred to.

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Differential operators and differential equations of infinite order with constant coefficients. Researches in connection with integral functions of finite order. By P. C. Sikkema. Groningen, Noordhoff, 1953. 4+223 pp. 11.50 florins; cloth 13.50 florins.

This monograph is essentially the author's Groningen thesis and describes the results of his researches into the following three general problems. Let $F(D)=\sum_{n=0}^{\infty} a_n D^n$, $D\equiv d/dx$, designate a differential operator of infinite order with constant coefficients, and let $F(D) \rightarrow y(x)$ designate the result of applying $F(D)$ to y, i.e., $F(D)\rightarrow y(x) = \sum_{n=0}^{\infty} a_n y^{(n)}(x)$. (1) Under what conditions does the expression $F(D)\rightarrow y(x)$ possess more than formal meaning? (2) What are the properties of the function $h(x)=F(D)\rightarrow y(x)$? (3) What can be said about the solutions $y(x)$ of the differential equation of infinite order $F(D)\rightarrow y(x)=h(x)$? The functions y and h are restricted in this work to be entire and of finite order, and the theorems obtained relate the properties of F with the order and type properties of y and h. Some of the results yield generalizations of theorems of H. Muggli, I. M. Sheffer, and J. M. Whittaker. The exposition is detailed, and workers in the field will find useful the exactitude with which the author has worked out the statements of his theorems as well as the numerous summaries of previously known results.

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The Michigan Mathematical Journal. Vol. 2, no. 1. University of Michigan Press, Ann Arbor, 1954. 94 pp. $2.00 per volume to individuals ordering directly, $4.00 to others.