THE JUNE MEETING IN VANCOUVER

The five hundred fifteenth meeting of the American Mathematical Society was held at the University of British Columbia, Vancouver, Canada, on Saturday, June 18, 1955, following the meeting on Friday of the Pacific Northwest Section of the Mathematical Association of America. Attendance was 71, including 54 members of the Society.

By invitation of the Committee to Select Hour Speakers for Far Western Sectional Meetings, Professor Iacopo Barsotti of the University of Pittsburgh and University of Southern California addressed the Society on *Algebraic group-varieties*. He was introduced by Professor R. M. Winger. Professor T. E. Hull presided at the session for contributed papers.

On Friday evening before the meeting there was a joint dinner of the Society and the Association, at which the visitors were greeted by Dean W. H. Gage of the University of British Columbia.

Following are the abstracts of papers presented at the meeting, those whose numbers are followed by "*t*" having been read by title. Mr. John was introduced by Professor Casper Goffman, Mr. Kobayashi by Professor C. B. Allendoerfer, Mr. Montague by Mr. Dana Scott, Dr. Saworotnow by Professor Choy-tak Taam, and Professor Wasel by Reverend T. J. Saunders.

ALGEBRA AND THEORY OF NUMBERS

596. R. A. Beaumont: *Free R-modules and algebras over a noncommutative ring.*

Let ${}_mR_n$ be the set of m by n matrices over a ring R with identity. A matrix A in ${}_mR_n$ is a unit if there exists a matrix B in ${}_nR_m$ such that $AB=I_m$ and $BA=I_n$. The only rings for which units exist $(m\neq n)$ are noncommutative rings which satisfy neither chain condition for right ideals or for left ideals. Every free R-module has a unique basis number if and only if there are no units in ${}_mR_n$ for every m, n, $m\neq n$. A necessary and sufficient condition is found for the equivalence of algebras over a noncommutative ring. This condition is in the form of a matrix identity involving the multiplication tables of the algebras. (Received May 5, 1955.)

597. D. G. Higman: *On integral representations of orders in separable algebras.*

J.-M. Maranda has recently established a very interesting set of results concerning representations of finite groups by groups of automorphisms of modules finitely generated over Dedekind rings (Canadian Journal of Mathematics vol. 5 (1953) pp. 344–355, and a second paper to appear soon in the same Journal). For representations Γ, Δ of the group ring $\mathfrak{O}$ by matrices in a Dedekind ring $\mathfrak{o}$, and $\mathfrak{a}$ an integral ideal in $\mathfrak{o}$, let $B(\Gamma, \Delta; \mathfrak{a})$ denote the $\mathfrak{o}$-module of binding systems, and let $B_0(\Gamma, \Delta; \mathfrak{a})$ denote the $\mathfrak{o}$-submodule of binding systems strongly equivalent to $\mathfrak{O}$ (see the first mentioned paper of Maranda for definitions). The present note is based on the observation that the essential property of the group ring for Maranda's work is the following: The annihilators in $\mathfrak{o}$ of the modules $B(\Gamma, \Delta; \mathfrak{a})/B_0(\Gamma, \Delta; \mathfrak{a})$ for all Γ, Δ, $\mathfrak{a}$, have a nonzero intersection—in fact the group order N is contained therein. Let A be a finite-dimensional algebra over the quotient field K of $\mathfrak{o}$, and let $\mathfrak{O}$ be an $\mathfrak{o}$-order in A. Assuming that $\mathfrak{O}$ has a linearly independent $\mathfrak{o}$-basis it is proved that the corresponding intersection is nonzero if and only if A is separable. This result suffices to extend Maranda's theory without essential change from finite groups to orders in separable algebras. (Received May 6, 1955.)

598*t*. D. G. Higman: *Remark on Frobenius algebras.*

A finite-dimensional associative algebra A with identity element over a field K is a Frobenius algebra if and only if there exists an invariant bilinear form on A, i.e., a bilinear for (a, b) on A with values in K which is nonsingular and such that $(ab, c) = (a, bc)$. If $a_1, \cdots, a_n$ is a basis of A, a dual basis $\bar{a}_1, \cdots, \bar{a}_n$ is defined by $(a_i, \bar{a}_j) = \delta_{ij}$. The following theorems are proved: *Theorem* 1. A representation Γ of A in K is a direct sum of principal indecomposable representations if and only if there exists a matrix X in K such that $\sum \Gamma(\bar{a}_i)X\Gamma(a_i) = I$, the identity matrix. *Theorem* 2. A is separable if and only if $c(A) =$ the center of A, where $c(a) = \sum \bar{a}_i a a_i$. (Received May 6, 1955.)

599*t*. S. A. Jennings: *Radical rings with nilpotent associated groups.*

Let R be an associative ring which is a radical ring in the sense of Jacobson. Then the elements of R form a group G under the multiplication $x * y = x + y + xy$. The group G is said to be associated with R, while the Lie ring $\Lambda = (R)_l$ formed by the elements of R under addition and the multiplication $x \circ y = xy - yx$ is the Lie ring associated with R. Relations between R, G, and Λ are studied. In particular, it is proved that if G is nilpotent then Λ is nilpotent, and conversely. (Received May 4, 1955.)

600*t*. S. A. Jennings and H. J. Zassenhaus: *Non-archimedean group valuations.*

Let ϕ be a non-archimedean valuation of a field F, let $\mathfrak{o}_\phi = \{\lambda \mid \phi(\lambda) \leqq 1\}$, $\mathfrak{p}_\phi = \{\lambda \mid \phi(\lambda) < 1\}$ where $\lambda \in F$, and let χ be the characteristic of $\mathfrak{o}_\phi$ modulo $\mathfrak{p}_\phi$. For any group G, a real valued non-negative function Φ on Γ, the group ring of G over F, will be called a G-invariant extension of ϕ if Φ is a Kürschak valuation of Γ such that $\Phi(g-1) < 1$, $\Phi(1) = 1$, $\Phi(\lambda x) = \phi(\lambda)\Phi(x)$ for $g \in G$, $\lambda \in F$, $x \in \Gamma$. A function ψ on G will be called a χ-valuation of G over F if $0 \leqq \psi(g) < 1$, $\psi(1) = 0$, $\psi(gh) \leqq \max(\psi(g), \psi(h))$, $\psi(ghg^{-1}h^{-1}) \leqq \psi(g)\psi(h)$, $\psi(g^n) = \psi(g)$ if $(n, \chi) = 1$, $\psi(g^\chi) \leqq \max(\phi(\chi)\psi(g), \psi(g)^\chi)$ and $\psi(g^\chi) = \phi(\chi)\psi(g)$ if $\psi(g)^{\chi-1} < \phi(\chi)$, g, $h \in G$. If Φ is a G-invariant extension of ϕ, then $\psi(g) = \Phi^*(g) = \Phi(g-1)$ is a χ-valuation of G, while any χ-valuation ψ of G induces a G-invariant extension $\Phi = \psi^*$ of ϕ by taking for all cases in which x can be represented as $\sum \lambda_{i_1, \cdots, i_r}(gi_1 - 1) \cdots (gi_r - 1)$, $\psi^*(x) = \inf \max \phi(\lambda i_1, \cdots, i_r)\psi(gi_1) \cdots \psi(gi_r)$ for $x = \sum \alpha_g g \in \Gamma$, and $\psi^*(x) = \phi(\alpha_1)$ otherwise. Relations between χ-valuations of G and G-invariant extensions of ϕ are studied. In particular, the subgroups $G_{\chi, \rho} = \{g \mid \Phi(g) \leqq \rho\}$ are identified abstractly for certain discrete G-invariant extensions of the trivial valuation $\phi(\lambda) = 1$, $\lambda \neq 0$ of a prime field F of characteristic χ. (Received March 28, 1955.)

601. J. E. Maxfield (p), R. G. Selfridge, and J. S. Lomont: *Similarity classifications of complex matrices.*

The following theorems, among others, are proved: A necessary and sufficient condition that an n by n matrix that is similar to a diagonal matrix be similar to a matrix A having only rational integral entries is that the characteristic roots of A be algebraic integers and that the traces of the first n powers of A be rational integers. A necessary and sufficient condition that an n by n matrix C that is similar to a diagonal matrix be similar to a real matrix is that the traces of the first n powers of C be real. A necessary and sufficient condition that a matrix be similar to a real matrix is that it be similar to its complex conjugate. (Received April 19, 1955.)

602t. E. V. Schenkman: *On the Engel condition of order 2 for groups.*

Let G be a group in which $[[b, a]a]$ is the identity for all a and b in G. Then for arbitrary a, b, c, d in G, $[[[a, b]c]d]$ is the identity and $[[a, b]c]$ has order 3. Thus G is nilpotent of class 3. (Received May 26, 1955.)

Analysis

603t. M. G. Arsove: *Normal families of subharmonic functions.*

Let $\mathcal{J}$ be a family of subharmonic (or more generally δ-subharmonic) functions w on a plane region Ω, let $\Phi_w^z(t)$ be the total variation of the mass distribution for w on the disc of radius t centered at z, let K be any compact subset of Ω, and let $\Psi_{\mathcal{J}}^K(r) = \sup_{w \in \mathcal{J}, z \in K} \int_0^r \Phi_w^z(t) d \log t$. Theorem: if the functions in $\mathcal{J}$ are locally uniformly bounded and (*) $\lim \sup_{r \to 0} \Psi_{\mathcal{J}}(r) < \infty$ for all compact subsets K of Ω, then (1) $\mathcal{J}$ is normal, (2) every $w \in \mathcal{J}$ is the difference of continuous subharmonic functions, and (3) the total variations of the mass distributions for functions in $\mathcal{J}$ are uniformly bounded on compact subsets of Ω. A sufficient condition for (*) is that the mass distributions for the functions w in $\mathcal{J}$ be given by density functions ρ_w forming a bounded subset of L^p, $1 < p \leqq \infty$. Under the norm $\|w\| = \|w\|_p + \|\rho_w\|_p$ the δ-subharmonic functions of finite norm comprise a Banach space, and for $p = 2$ normality considerations lead to the existence of a reproducing kernel analogous to the Bergman kernel function for L^2 spaces of analytic or harmonic functions. (Received May 11, 1955.)

604. Arthur Erdélyi and C. A. Swanson (p): *Asymptotic forms for Whittaker functions.* Preliminary report.

Uniform asymptotic representations for the Whittaker functions $M_{k,m}(4kx)$ and $W_{k,m}(4kx)$ are obtained for unrestricted real x and large values of $|k|$, under suitable restrictions upon arg k. The method used is an adaptation of T. M. Cherry's technique for treating differential equations with transition points; solutions of the confluent hypergeometric equation are compared with appropriate Airy functions and Bessel functions by means of (singular) Volterra integral equations, the solutions being identified by their asymptotic behavior as x approaches a singularity while k is fixed. Two asymptotic forms are obtained for each of the two functions $M_{k,m}(4kx)$ and $W_{k,m}(4kx)$. One of these involves Bessel functions of order $2m$ and holes in the interval $-\infty < x \leqq 1 - \epsilon < 1$ as $k \to \infty$; the other involves Airy functions and holds in the interval $0 < \epsilon \leqq x < \infty$ as $k \to \infty$. In addition, the Bessel function approximation for $M_{k,m}$ holds also for fixed k as $x \to 0$, and the Airy function approximation for $W_{k,m}$ holds also for fixed k as $x \to \infty$. (Received May 5, 1955.)

605. R. T. Ives: *Semi-convexity and products of locally bounded spaces.*

A set U containing the origin of a linear space is called *semi-convex* iff there is a $\beta\in]0, 1]$ such that for all non-negative λ_1 and λ_2 with $\lambda_1+\lambda_2=\beta$, it is true that $\lambda_1 U+\lambda_2 U\subset U$. For such a β, U is called *β-convex.* If τ is an admissible quasi-norm for a locally bounded topological linear space E, and $U=\{x\in E: \tau(x)<1\}$, then U is β-convex iff $\beta\tau(x+y)\leqq\tau(x)+\tau(y)$ for all $x\in E$, $y\in E$. A topological linear space is called *locally semi-convex* iff it has a fundamental system of semi-convex neighborhoods of the origin. A space is locally semi-convex iff it is isomorphic to a subspace of a product of locally bounded spaces. The space of equivalence classes of real measurable functions on the unit interval, metrized by $\rho=[\int_0^1|f-g|]/[1+\int_0^1|f-g|]$, is not locally semi-convex; the same for an example of Bourgin. This answers a question of Klee [Trans. Amer. Math. Soc. vol. 78 (1955) p. 34]. (Received May 5, 1955.)

606*t*. Jacob Korevaar: *Distributions defined by fundamental sequences.* IV. *The integral of a product and Schwartz' theory.*

Let the distribution ϕ be integrable over $[a, b]$, and let the product ϕg be defined on $(a-\epsilon, b+\epsilon)$ for some $\epsilon>0$. Then $\int_a^b\phi g$ exists and may be computed by integration by parts. Example $(a<\tau<b)$: $\int_a^b\delta_\tau(t)g(t)dt=[U_\tau(t)g(t)]_a^b-\int_a^b U_\tau(t)g'(t)dt=g(\tau)$. Hadamard's finite part of a certain kind of divergent integral is the integral of a product ϕg. One also has $\int_a^b\phi g=\lim\int_a^b f_n g$ for integrable $\{f_n\}$ such that $f_n\rightarrow\phi$ on $(a-\epsilon, b+\epsilon)$. Yield: a result on the (ordinary) convergence of Fourier series. A result on substitution in a distribution is proved. Example: when $g'(t)\neq 0$, $g(\tau)=0$, then $\delta\{g(t)\}=|g'(\tau)|^{-1}\delta_\tau(t)$. Application is made to substitution in a definite integral. Let $\mathfrak{D}$ denote the linear space of all $g\in C^\infty[0, \infty)$ which vanish for $t\geqq c_g$ where c_g varies with g. In $\mathfrak{D}$ the relation $g_n\rightarrow g$ will mean that (i) for some $c>0$, $g(t)=g_n(t)=0$ for $t\geqq c$, $n=1, 2, \cdots$; (ii) for every $k\geqq 0$, $g_n^{(k)}(t)\rightarrow g^{(k)}(t)$ uniformly on $[0, c]$. The inner product of a distribution ϕ and a function $g\in\mathfrak{D}$ is defined as $\int_0^b\phi g$ $(b>c_g)$. It is shown that there is a one to one correspondence between the distributions ϕ on the half-line $t\geqq 0$ and the continuous linear functionals T on $\mathfrak{D}$ (Schwartz' distributions) such that if $\phi\leftrightarrow T$, then $(\phi, g)=T(g)$ for every $g\in\mathfrak{D}$. The correspondence is preserved under operations such as differentiation. (Received April 20, 1955.)

607*t*. Jacob Korevaar: *Distributions defined by fundamental sequences.* V. *Fourier series of distributions.*

Corresponding to every distribution ϕ there will be infinitely many Fourier series (1) $a_0/2+\sum_1^\infty(a_k\cos kt+b_k\sin kt)$ on the interval $[0, 2\pi]$. When ϕ is integrable over $[0, 2\pi]$, then one of the series is given by a_k $(b_k)=(1/\pi)\int_0^{2\pi}\phi(t)\cos(\sin) ktdt$. The following results are needed in applications. A series (1) is a Fourier series of a distribution if and only if there are constants M and m such that $|a_k|+|b_k|<Mk^m$, $k=1, 2, \cdots$. (This condition ensures convergence on $(0, 2\pi)$ in the sense of distributions.) Termwise differentiation of a Fourier series for ϕ gives a Fourier series for ϕ'. Every Fourier series of ϕ converges to ϕ on $(0, 2\pi)$. Some definitions follow. A sequence of integrable functions $f_n(t)$ converging to a distribution ϕ on $(0, 2\pi)$ is called F-admissible when (i) the limits $\lim (1/\pi)\int_0^{2\pi}f_n(t)\cos(\sin) ktdt=a_k$ (b_k) exist for $k=0, 1, \cdots$; (ii) the a_k and b_k satisfy a set of inequalities (2). Let $\{f_n\}$ be one of the infinitely many F-admissible sequences converging to a given ϕ on $(0, 2\pi)$. The Fourier series of $\phi\{f_n\}$ is the series (1) formed with the coefficients (i). It converges to ϕ on $(0, 2\pi)$. (Received April 20, 1955.)

608t. Erwin Kreyszig: *Partial differential equations with singular coefficients.*

Particular solutions (1) $u(z, z^*) = \int_{-1}^{1}(z/2)^n \exp\left(\sum_{\mu=0}^{m} q_\mu(z, z^*)t^\mu\right)(1-t^2)^{n-1/2}dt$ of partial differential equations (2) $u_{zz^*}+B(z, z^*)u_{z^*}+C(z, z^*)u=0$ satisfy also an ordinary differential equation (3) $\sum_{k=0}^{\kappa} b_\kappa(z, z^*)\partial^n u/\partial z_1^\kappa=0$, $b_k=1$, $z=z_1+iz_2$, $z^*=z_1-iz_2$, of order $k=m+1$. See Bergman, Rec. Math. (Mat. Sbornik) vol. 2 (1937) p. 1169 and the previous notes of the author. All types of (2) corresponding to (1) can be determined. Properties of (1) can be obtained using the theory of ordinary differential equations. B and C might have singularities which then correspond to those of the coefficients b_κ of (3). The fact that B and C depend in a simple manner on q_0, q_1, and q_2 only is advantageous for obtaining relations between B and C and the quantities b_κ. Theorems: I. If B has a pole of order $p \geqq 1$ at a point $z=a$, then $b_\kappa=b_\kappa(m)$ has a pole of order ph, $h=m+1-\kappa$, at the same point. If B has in particular a pole of first order at $z=a$, then (3) is of Fuchs type at this point. II. If C has a pole of order $j>1$ at a point $z^*=c^*$, then $b_\kappa(m)$ has a pole of order $h(j+1)/2+(\delta_{0\kappa}+\delta_{2\kappa})(j-1)/2$ at the same point; $\delta_{0\kappa}$ and $\delta_{2\kappa}$ are Kronecker symbols. It is possible to generalize the methods for other types of particular solutions (1). (Received April 25, 1955.)

609t. Anneli L. Lax: *On Cauchy's problem for partial differential equations with multiple characteristics.*

Let $L[u]=0$ be a linear partial differential equation of order m for $u(x, t)$, a_{ij} the coefficient of $\partial^{i+j}/\partial x^i\partial t^j$ in $L[u]$. Associate $p(z)=a_{0,m}z^m+a_{1,m-1}z^{m-1}+\cdots+a_{m,0}$ with the principal part of $L[u]$ and $q_k(z)=a_{0,k}z^k+a_{1,k-1}z^{k-1}+\cdots+a_{k,0}$ with the homogeneous operator of order k in $L[u]$. For constant coefficients a_{ij}, Cauchy's problem for a noncharacteristic initial curve is well posed if and only if the roots of $p(z)$ are real, and, whenever $z=\lambda$ is an r-tuple root of $p(z)$, it is an $(r-s)$-tuple root of $q_{m-s}(z)$, $s=1, 2, \cdots, r-1$. For variable coefficients $a_{ij}(x, t)$ and one multiple characteristic, $L[u]$ is transformed so that the multiple characteristic coincides with the t-axis. Cauchy's problem is well posed if for the transformed equation, the roots of $p(z)$ are real functions and, whenever 0 is an r-tuple of $p(z)$, it is an $(r-s)$-tuple root of $q_{m-s}(z)$, $s=1, 2, \cdots, r-1$. The results for constant a_{ij} are proved by using an inequality from the theory of solutions by Fourier series and applying the theory of algebraic functions, those for variable a_{ij} by a reduction of $L[u]$ to an equivalent first order non-degenerate hyperbolic system. (Received April 25, 1955.)

610. A. E. Livingston (p) and Lee Lorch: *The zeros of certain sine-like integrals.*

Let (i) $f(t) \geqq 0$, $0 \leqq t<1$, (ii) $f(t) \not\equiv 0$ on any subinterval of $[0, 1]$, (iii) $f(t+n) = (-1)^n f(t)$, $n=1, 2, \cdots$, (iv) $f(t)/t$ be Lebesgue integrable over $(0, 1)$. Let C be defined (uniquely) by the two conditions $2\int_0^C f(t)dt=\int_0^1 f(t)dt$ and $0<C<1$. Then the function $\int_x^\infty f(t)dt/t$ has precisely one zero in the interval $n<x<n+1$, $n=0, 1, \cdots$, say z_n, and we show that $z_n-n \downarrow C$. A by-product of the proof is that the function $tG(2t)=t[\psi(t+1/2)-\psi(t)]$ is completely monotonic, $0<t<\infty$, where $\psi(t)=\Gamma'(t)/\Gamma(t)$. (Received May 5, 1955.)

611. M. D. Marcus and B. N. Moyls (p): *Second order Hermitian systems of linear differential equations.*

Let A, B, C be Hermitian n-square matrices. The behavior of solutions of the vector-matrix differential equation $A\ddot{x}(t)+B\dot{x}(t)+Cx(t-k)u(t-k)=F(t)$ is determined by the roots of (1) $|As^2+Bs+\exp(-ks)C|=0$. $u(t-k)$ is the unit step function, $k\geqq 0$. Considered here is the more general (2) $|s^2A+sB+p(s)C|=0$. Let s_1, s_2, p_1, p_2 be respectively the real and imaginary parts of s and $p(s)$. Results: Assume A, B, $C\geqq 0$. (i) Let $\inf p_2(s_1, s_2)s_2^{-1}=d$ for $s_2\neq 0$, $s_1>0$; $\inf p_1(s_1, 0)s_1^{-1}=h$ for $s_1>0$. If either $B+qC>0$ for $q=d$, h or $B+qC\geqq 0$ when $A>0$, then all roots of (2) lie in the negative half-plane $\operatorname{Re} s\leqq 0$. (ii) If $p=\exp(-ks)$, $k>0$, the hypotheses of (i) become $B-kC>0$, $B>0$ or $B-kC\geqq 0$, $A>0$. (iii) If $p=1$ the condition is $A>0$ or $B>0$. In case the hypotheses of (i) fail, vertical strips in the right half-plane are given in which the roots of (2) must lie. (Received April 18, 1955.)

612t. Leo Moser and Max Wyman: *Asymptotic formula for the Bell numbers.*

Properties of the Bell numbers G_n, defined by $e^{e^z-1}=\sum_{n=0}^{\infty} G_nz^n/n!$ have been studied by many authors. However, only L. F. Epstein [Journal of Mathematics and Physics vol. 18 (1939) pp. 166–182] has given a formula for the asymptotic behavior of G_n. His method is rather long and yields only the first term of the asymptotic formula. Moreover, his final result contains an error. In the present paper it is shown by an entirely different method that $G_n\sim(R+1)^{-1/2}e^{n(R+R^{-1}-1)}$ where R is the real solution of $Re^R=n$. Furthermore, the complete asymptotic expansion of G_n is obtained. (Received March 21, 1955.)

613t. Edgar Reich: *On schlicht functions with real coefficients.* Preliminary report.

Let S_r be the class of functions $f(z)=z+a_2z^2+\cdots$ (a_k real) which are regular and schlicht in the unit circle. It is desirable to obtain distortion theorems for S_r which become false when either the hypothesis that the a_k are real is dropped, or when S_r is replaced by the class of typically real functions. An example of a theorem of this type is $|\operatorname{Im}[1/f(z)]|\leqq|z|+1/|z|$ (obtained by integrating Tammi's differential equation [Ann. Ac. Sci. Fenn. AI, 1954, paper no. 173]); the result is sharp in the sense that the right-hand side cannot be replaced by any smaller function of $|z|$. (Received April 20, 1955.)

614t. P. P. Saworotnow: *On the imbedding of a right complemented algebra into Ambrose's H^*-algebra.* Preliminary report.

A right complemented algebra is a Banach algebra $\mathfrak{A}$ which is a Hilbert space and which has the property that every orthogonal complement of a right ideal is a right ideal. This notion was introduced in the author's dissertation (Harvard University, 1954) (see also Bull. Amer. Math. Soc. Abstract 61-2-286). We shall say that an element x in $\mathfrak{A}$ has a left adjoint x^l if $(xy, z)=(y, x^lz)$ holds for every y, z in $\mathfrak{A}$ (compare with W. Ambrose, Trans. Amer. Math. Soc. vol. 57, pp. 364–368). The author proves that every semi-simple right complemented algebra contains a dense set of elements having a left adjoint. It follows then that every semi-simple right complemented algebra is also left complemented. The other more interesting result: for every semi-simple right complemented algebra $\mathfrak{A}$ it is possible to construct an H^*-algebra $\tilde{\mathfrak{A}}$ such that $\mathfrak{A}$ is dense in $\tilde{\mathfrak{A}}$, i.e., every simple semi-simple right complemented algebra is of the type described in one of the examples given in the author's dissertation. (Received April 8, 1955.)

615*t*. J. B. Serrin: *On the Harnack inequality for second order elliptic differential equations.*

The author considers the linear elliptic differential operator in n variables, $L=\sum a_{ij}\partial^2/\partial x_i\partial x_j+\sum b_i\partial/\partial x_i+c$, where the coefficients are functions of $x=(x_1, \cdots, x_n)$. Concerning the coefficients he assumes merely the normalization det $(a_{ij})=1$ and the conditions (1) $\lambda(x)=$least proper value of $(a_{ij})\geqq k>0$, (2) $|b_i|\leqq B<\infty$, (3) $c\leqq 0$, and (4) $|a_{ij}(x)-a_{ij}(y)|\leqq\phi(|x-y|)$, where $\int_0\phi(r)r^{-1}dr<\infty$. Let U be a non-negative function satisfying $LU=0$ in a bounded domain T, and let R be a closed region in T. Let x and y be any two points in R. Then there exists a positive constant M, depending only on R, T, k, B, and $\int_0^d\phi(r)r^{-1}dr$ ($d=$diameter of T), such that $M^{-1}U(y)\leqq U(x)\leqq MU(y)$. The proof is elementary in character and based on the maximum principle; moreover, the constant M, although complicated, may be given explicitly (this is important for several applications we have in mind). It should be noted that previous proofs of Harnack's inequality for elliptic equations (cf. Lichtenstein, Rend. Circ. Mat. Palermo vol. 33 (1912), and Feller, Math. Ann. vol. 102 (1930)) involve considerably stronger assumptions on the coefficients, and resort to the existence of a Green's function; furthermore they supply only the existence of M, while we give an explicit value. (Received April 22, 1955.)

616*t*. A. D. Wasel: *An integral operator in the theory of compressible fluids.*

The author considers (1) $\psi^*_{z\bar z}+F\psi^*=0$ (Bergman, Trans. Amer. Math. Soc. vol. 62 (1947) p. 464) which arises in the theory of compressible fluids when studying flows in the pseudo-logarithmic plane, i.e., the plane whose coordinates are $\lambda=(z+\bar z)/2$, $\theta=(z-\bar z)/2i$. λ is a function of the speed, see (2.7) of the above paper, and θ the angle which the velocity vector forms with the positive x-axis of the physical plane. Theorem. Let $R_n(\lambda)$, $n=0, 1, \cdots$, satisfy $R'_{n+1}=R''_n+4FR_n$, $R_0(\lambda)=1$ ($R'_n\equiv dR_n/d\lambda$), $R_n(-\infty)=0$, and let $\sum_{n=0}^\infty a_nz^n$ be the function element of $f(z)$ with radius of convergence r. Then $R^{[1]}(f)=\lim_{N\to\infty}\int_C\sum_{n=0}^N R_n(\lambda)(t^2z)^nf(t^2z)t^{-1}\exp(-1/(2t^2))dt$, C a simple path joining -1 and 1 and avoiding the origin, is a solution of (1). $R^{[1]}$ is defined in $D=[(|z|<2|\lambda|)\cap(|z|<r)]$. By substituting $u=t^2$ and taking residues, $R^{[1]}$ becomes $\lim_{N\to\infty}\ \pi i[a_0\sum_{n=0}^N\ (-)^nR_nz^n/(n!2^n)+a_1z\sum_{n=0}^N\ (-)^nR_nz^n/((n+1)2^{n+1})+\cdots]\equiv\lim_{N\to\infty}\sum_{n=0}^\infty a_nS_n^{(N)}(z)$. A sequence of functions Q_n exists which dominate R_n (v. Mises and Schiffer, Adv. in Applied Mech., 1948, pp. 258–259). Further $|S_n^{(N)}|\leqq|S_0^{(N)}|$, $n=1, 2, \cdots$. These facts imply that $R^{[1]}$ converges in D. Direct substitution shows that each $\lim_{N\to\infty}S_n^{(N)}$ is a solution of (1) and the theorem follows. (Received April 8, 1955.)

617*t*. C. H. Wilcox: *A generalization of theorems of Rellich and Atkinson.*

An exterior region V is a region consisting of all points outside a closed bounded surface S. A function $u(r)$ is a radiation function for V if it is of class C^2 and satisfies the Helmholtz equation and Sommerfeld's radiation condition $\lim_{R\to\infty}\int_{r=R}|\partial u/\partial r-iku|^2dS=0$ in the closure of V, where $k\neq 0$ is a complex number satisfying Im $k\geqq 0$. It is shown that every radiation function $u(r)$ satisfies $\int_\Omega|u|^2d\Omega=O(1/r^2)$, $r\to\infty$, where r, θ, ϕ are spherical coordinates, Ω is the unit sphere and $d\Omega$ the element of solid angle. As a consequence it is shown that $u(r)=(e^{-ikr}/r)\sum_{n=0}^\infty f_n(\theta, \phi)/r^n$ in $r\geqq c$, provided $r=c$ contains S, where the series converges absolutely and uniformly in $r\geqq c+\epsilon>c$. This result was proved by F. V. Atkinson [Philosophical Magazine vol.

40 (1949) pp. 645–651] under the additional hypothesis that $u(r)=O(1/r)$, $r\rightarrow\infty$. With the help of the expansion the following uniqueness theorem is proved. Let V be an exterior region bounded internally by a regular surface S (in the sense of Kellogg, *Foundations of potential theory*, p. 112) and let f be an arbitrary continuous function on S. Then there is at most one radiation function $u(r)$ for V which satisfies $u=f$ or $\partial u/\partial n=f$ on S. This result was proved by F. Rellich [Jber. Deutschen Math. Verein. vol. 53 (1943) pp. 57–65] on the assumption that $k>0$. (Received May 3, 1955.)

Applied Mathematics

618t. Isadore Heller: *Neighbor relations on the convex of cyclic permutations.*

Two vertices of a polyhedron are called neighbors of order k when they have a face of dimension k, and none of lower dimension, in common. $K(P)$ denotes the maximum value of k for a given polyhedron P, and $[X]$ denotes the largest integer not exceeding X. For the convex hull (polyhedron) P_n of all permutations of n elements (represented by square matrices of order n and interpreted as points in n^2-space) it was shown (Bull. Amer. Math. Soc. (1955) and Proceedings of the January 1955 Symposium on Linear Programming) that $K(P_n)=[n/2]$, which is rather small as compared with dim $P_n=(n-1)^2$. For the convex hull Q_n of all cyclic permutations of n elements that leave no element fixed, H. Kuhn performed computations showing that any two vertices of Q_5, but not any two vertices of Q_6, are neighbors of order 1, which means that $K(Q_5)=1$ and $K(Q_6)>1$. The present note, dealing with general n, proves, for $n\geqq 8$: (1) $K(Q_n)=K(P_n)-1=n/2-1$ if $n=4m+2$; (2) $K(Q_n)=K(P_n)=[n/2]$ if $n\neq 4m+2$. For $n=1, 2, \cdots, 6, 7$, $K(Q_n)=0, 0, 1, 1, 1, 2, 2$ respectively. (Received May 5, 1955.)

619t. Erwin Kreyszig: *Inclusion of eigenvalues.*

Given a Hermitian matrix A and an iterative sequence of vectors x_0, $x_1=Ax_0, \cdots$, $x_s=A^s x_0$. To determine intervals of the real axis which contain at least one of the eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ of A ("inclusion intervals"). According to an (unpublished) theory of H. Wielandt it is possible to use all of the given vectors for this purpose. The accuracy of the inclusion depends on the number of the vectors used; but the complexity of the computation increases with the number of the vectors used. For practical purposes the "two step method" (i.e. the determination of inclusion intervals by aid of three vectors x_{s-2}, x_{s-1}, x_s) seems to be the optimum: The inclusion is much better than those obtained by classical methods (Weinstein, Proc. Nat. Acad. Sci. U.S.A. vol. 20 (1934) p. 529; Collatz, *Eigenwertaufgaben*, Leipzig, 1949); on the other hand the numerical computation involved is not too complicated. It is possible to determine an inclusion interval of minimal length. One can always determine three inclusion intervals the intersections of which are only the end points. The analytic procedure involves differences of great numbers; the accuracy of the numerical computation can be improved by aid of orthogonalization methods similar to those used by Karush (Pacific Journal of Mathematics vol. 1 (1951) p. 233). (Received April 25, 1955.)

620t. Erwin Kreyszig: *Improved two step inclusion methods for eigenvalues.*

The inclusion of eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ of an Hermitian matrix A by aid of two step methods (see the previous note) can be improved using special information, e.g. on eigenvalue-free intervals, perhaps in the case of positive definite A or on bounds for certain eigenvalues. This information can be derived from physical properties of the problem or using general methods (Collatz, Zeits. Angew. Math. Mech. vol. 19 (1939) p. 224). There is no principal difference between the two step method and classical methods, but the inclusion is much better than in the case of the latter. Two step methods can be used also in the case of ordinary differential equations and integral equations. It is possible to develop geometrical procedures which yield a general picture of the inclusion intervals. When analytical processes for obtaining the inclusion intervals are used approximation methods are advantageous for lowering the number of computations. Orthogonalization can be used in the same manner as in the case without special information. (Received April 25, 1955.)

621. W. M. Stone: *On the probability of detection with a postdetection filter.*

The theory of Kac and Siegert (Journal of Applied Physics vol. 18 (1947) pp. 383–397) for radio receivers has been extended to realistic systems involving first and second order filters. Surprisingly simple formulas for probability of detection are obtained if signal amplitude is assumed to be Rayleigh-distributed. This is possible because certain infinite products and infinite sums involving the zeros of Bessel functions may be evaluated in closed form. The case of constant amplitude signal may be treated by a Gram-Charlier type of expansion since general expressions for the central standard moments of the distribution of output are readily obtainable. (Received May 9, 1955.)

Geometry

622. V. L. Klee, Jr.: *Strict separation of convex sets.*

The convex subsets A and B of a topological linear space are said to be *strictly* separated by a hyperplane H provided A is contained in one of the *open* half-spaces determined by H and B is contained in the other. This paper gives several results related to strict separation of convex sets. The basic theorem is (I): *For a locally compact closed convex subset A of a locally convex topological linear space E, the following two assertions are equivalent*: (i) *Whenever H is a hyperplane in E which supports A, then $A \cap H$ is linearly bounded.* (ii) *Whenever B is a closed convex subset of $E \backslash A$ which can be separated from A, then B can be separated from A by a hyperplane which misses A.* Among the corollaries is (II): *If A and B are disjoint closed convex subsets of E^n and neither contains a ray in its boundary, then A and B can be strictly separated by a hyperplane.* (Received May 2, 1955.)

623*t*. Shôshichi Kobayashi: *Affine transformation group of Riemannian manifolds.*

Let M be a complete irreducible Riemannian manifold. Then every transformation of M which preserves the affine connection associated with the metric is necessarily an isometry, except in the case where M is the 1-dimensional Euclidean space. The proof is divided into two cases: (1) If the transformation has no fixed point, then it is an isometry without exception and (2) if it has a fixed point, then it is an isometry except in the case mentioned above. (Received April 26, 1955.)

624. T. G. Ostrom: $n+1$ *curves, dualities, and Desargues' Theorem.*

An $n+1$ curve in a finite projective plane is a set of $n+1$ points, no three of which are collinear. Here $n+1$ denotes the number of points on a line. The following aspects of $n+1$ curves are investigated: (1) properties of collineations which leave an $n+1$ curve invariant. (2) ways in which an $n+1$ curve resembles a conic in a Desarguesian plane. (3) the relation between polarities and Fano's configuration when n is even. (4) applications to cyclic planes. (Received May 2, 1955.)

Logic and Foundations

625*t*. Richard Montague: *Well-founded relations; generalizations of principles of induction and recursion.*

For formalization see abstract 627. If α is a variable and $\phi(\alpha)$ a formula, then $\mathbf{O}\alpha\phi(\alpha)$ abbreviates $\mathbf{V}\alpha[\phi(\alpha) \wedge \mathbf{\Lambda}\beta[\phi(\beta)\rightarrow\beta=\alpha]]$, where β is the first variable different from α and not occurring in $\phi(\alpha)$. Generalizations of principles (I) and (II) of abstract 627 can be obtained using the notion of a well-founded relation (see Zermelo, Fund. Math. vol. 25 (1935)). (I) (*The general principle of induction*). *If the formula* $\phi(x)$ *does not contain the variable* y, *then the following is a theorem*: R *is well-founded* $\wedge\ \mathbf{\Lambda}y[y\in \mathit{field}\ R \wedge \mathbf{\Lambda}x[xRy\rightarrow\phi(x)]\rightarrow\phi(y)]\rightarrow\mathbf{\Lambda}y[y\in \mathit{field}\ R\rightarrow\phi(y)]$. (II) (*A general principle of definition by recursion*). *If the term* ζ *does not contain the variable* f, *then the following is a theorem*: R *is well-founded* $\rightarrow \mathbf{O}f$ [f *is function* $\wedge$ *domain* $f=\mathit{field}\ R \wedge \mathbf{\Lambda}x[x\in \mathit{field}\ R\rightarrow f(x)=\zeta(\{y \mid \mathbf{V}z[zRy \wedge y=f(z)]\})]$]. Recursion theorems for natural numbers, transfinite ordinals, and metamathematics, as well as (II) of abstract 627, can be obtained immediately from (II). (III) There is a term ζ not containing the variable f such that the converse of the theorem mentioned in (II) is a theorem. Using (II) an isomorphism theorem for well-founded relations can be obtained; a special case was proved in Mostowski, Fund. Math. vol. 36 (1949) pp. 143 ff. (IV) R *is well-founded*$\rightarrow\mathbf{V}x[R$ *is isomorphic to* $\mathbf{E}x]$. ($\mathbf{E}x$ is the membership relation restricted to the set x.) (Received May 6, 1955.)

626*t*. Dana Scott: *Definitions by abstraction in axiomatic set theory.*

For terminology and formalization see abstracts 627 and 628. The following is obtained without using the axiom of choice: (I) *If the variables* x, y, *and* z *are not bound in the formula* $\phi(x, y)$, *and the formulas* $[\phi(x, y)\rightarrow\phi(y, x)]$ *and* $[\phi(x, y) \wedge \phi(y, z)\rightarrow\phi(x, z)]$ *are theorems, then there is a term* $\zeta(x)$ (*whose free variables are those variables other than* y *which are free in* $\phi(x, y)$) *such that the following are theorems*: (i) $\mathbf{V}y\phi(x, y)\rightarrow\mathbf{V}y y\in\zeta(x)$, (ii) $\mathbf{\Lambda}y[y\in\zeta(x)\rightarrow\phi(x, y)]$, (iii) $\mathbf{V}z\phi(x, z)\rightarrow[\zeta(x)=\zeta(y)\leftrightarrow\phi(x, y)]$; in fact, the term obtained by eliminating defined symbols from $\{y \mid \phi(x, y) \wedge \sim\mathbf{V}z[\phi(x, z) \vee \rho z\in\rho y]\}$ will serve as $\zeta(x)$. Applications: It is customary to identify the cardinal number of a set with the least ordinal with which the set is cardinally equivalent; thus the axiom of choice is required to show that every set has a cardinal. But let $\phi(x, y)$ be a formula expressing the cardinal equivalence of x and y. Define the cardinal of x as $\zeta(x)$, where $\zeta(x)$ is the term mentioned in (I). The properties corresponding to (i) and (iii) are sufficient for an adequate notion of cardinal number. Thus the axiom of choice is unnecessary in this context. Similar remarks apply to relation types and isomorphism types. (Anthony P. Morse has introduced these latter notions into axiomatic set theory using the axiom of choice.) (Received May 6, 1955.)

627*t*. Alfred Tarski: *General principles of induction and recursion in axiomatic set theory.*

For logical symbols see Tarski, Mostowski, Robinson, *Undecidable theories*. Present remarks apply to formalized Zermelo-Fraenkel set theory, but extensions to other forms of axiomatic set theory are possible. Two kinds of expressions are distinguished —terms and formulas. The simplest terms are variables; if ζ, η are terms, then $\zeta \in \eta$, $\zeta = \eta$ are formulas; if $\phi(\alpha)$ is a formula and α a variable, then $\{\alpha | \phi(\alpha)\}$ is a term (denoting the set of sets α which satisfy $\phi(\alpha)$, if there is such a set, and the empty set otherwise); if ϕ, ψ are formulas, so are $[\phi \vee \psi]$, $\sim\phi$, etc. $\mathbf{C}x$ is the transitive closure of the set x; see Bernays, J. Symbolic Logic vol. 7 (1942) p. 136. Using transitive closure, one proves the set-theoretical principles of induction (I) and definition by recursion (II). (I) *If the formula $\phi(x)$ does not contain the variable y, then $\bigwedge y[\bigwedge x[x \in y \rightarrow \phi(x)] \rightarrow \phi(y)] \rightarrow \bigwedge y \phi(y)$ is a theorem.* (II) *If $\zeta(x)$ is a term not containing the variables y or z, then there is a term $\eta(x)$ with the same free variables as $\zeta(x)$ such that $\bigwedge x \eta(x) = \zeta(\{y | \bigvee z[z \in x \wedge y = \eta(z)]\})$ is a theorem.* Under the hypotheses of (II) it can also be shown there is a term $\lambda(x)$ where $\bigwedge x \lambda(x) = \zeta(\{y | \bigvee z[z \in \mathbf{C}x \wedge y = \eta(z)]\})$ is a theorem. If two terms $\eta(x)$ and $\eta'(x)$ satisfy the conclusion of (II), then $\bigwedge x \eta(x) = \eta'(x)$ is a theorem. (II) was formulated but not proved by the author; independent proofs were found by Richard Montague and Dana Scott. (Received May 6, 1955.)

628t. Alfred Tarski: *The notion of rank in axiomatic set theory and some of its applications.*

For terminology and formalization see abstract 627. By (II) of that abstract, recursive definitions of the form $Fx = \zeta(\{y | \bigvee z[z \in x \wedge y = Fz]\})$ can be replaced by explicit definitions. The rank of a set (introduced by a more involved method in Bernays, J. Symbolic Logic vol. 13 (1948) p. 67) can be defined recursively: $\rho x = \mathbf{C}\{y | \bigvee z[z \in x \wedge y = \rho z]\}$ or equivalently $\rho x = \{y | \bigvee z[z \in \mathbf{C}x \wedge y = \rho z]\}$. Application of rank: (I) *The theory of ordinals can be developed within the theory of ranks.* The following definition is equivalent to usual definitions of ordinals: x *is ordinal* $\leftrightarrow \rho x = x$. The ordinal sum of arbitrary sets x and y, defined recursively as follows: $x + y = x \cup \{w | \bigvee z[z \in y \wedge w = x + z]\}$, reduces, when x, y are ordinals, to the usual ordinal addition. (II) *If the formula $\phi(x)$ does not contain the variable y, then there is a term ζ (whose free variables are those variables other than x which are free in $\phi(x)$) such that the following are theorems*: (i) $\bigvee x \phi(x) \rightarrow \bigvee x x \in \zeta$, (ii) $\bigwedge x[x \in \zeta \rightarrow \phi(x)]$, (iii) $\sim \bigvee y \bigwedge x[x \in y \leftrightarrow \phi(x)] \rightarrow \zeta$ *is infinite.* (III) *It can be shown that, for any set x, the set of all sets y such that $\rho y \in \rho x$ exists*; let $\mathbf{R}x$ be this set. Every set is an element of some set $\mathbf{R}x$, where x is an ordinal; further, the sets $\mathbf{R}x$ are well-ordered by the inclusion relation. Thus the universe is, so to speak, effectively covered by a well-ordered sequence of sets. (Received May 6, 1955.)

Statistics and Probability

629. Melvin Dresher: *A moment inequality.*

In a previous paper [*Moment spaces and inequalities*, Duke Math. J. vol. 20 (1953) pp. 261–271] the author described a general procedure for obtaining integral inequalities by analyzing the boundary of the convex hull of a given curve. The present paper applies these techniques to derive the inequality max $[(\mu_1 - \mu_2)/(1 - \mu_1), \mu_1 - \sigma, 1 - 2\mu_1] \leqq \text{Median} \leqq \min [\mu_2/\mu_1, \mu_1 + \sigma, 2\mu_1]$, where μ_1, μ_2, σ are the first moment, second moment, and standard deviations, respectively, of the distribution function on the interval (0, 1). (Received May 4, 1955.)

630t. P. W. M. John: *On the Feller-Lundberg phenomenon in birth and death processes.*

For a stationary birth and death process let t_∞ be the time at which the population is infinite. Then t_∞ is a random variable with expected value $\bar{t}_\infty$. The Feller-Lundberg phenomenon asserts that under certain conditions the probability $p[t_\infty < \infty] > 0$. It is shown here that $p[t_\infty < \infty] > 0$ implies $p[t_\infty < \infty] = 1$. Indeed, an expression is found for $\bar{t}_\infty$, from which the above result, as well as a new proof of the Feller-Lundberg Theorem, extended to birth and death processes, is derived. The latter result differs in detail from one obtained by Reuter and Ledermann. (Proc. Cambridge Philos. Soc. vol. 49 (1953) pp. 256–262.) (Received March 28, 1955.)

Topology

631*t*. Ernest Michael: *On a theorem of Kuratowski.*

The following theorem, due to Kuratowski in the separable case, has been proved by J. Dugundji (unpublished). If X and Y are metric spaces, if Y is LC^n, and if $A \subset X$ is closed with (covering-) dim $(X-A) \leqq n+1$, then every continuous $f: A \rightarrow Y$ can be extended continuously over an open $U \supset A$; if Y is also C^n, then one can even take $U = X$. Similar methods, together with the relevant known extension theorems without dimensional assumptions (cf. O. Hanner [Ark. Mat. vol. 1 (1951) pp. 375–382], and C. H. Dowker [Ark. Mat. vol. 2 (1952) pp. 307–313]) imply that this result remains true for all normal (resp. collectionwise normal) X if and only the metric space Y is complete and separable (resp. complete). (In general, one need only require that dim $(C) \leqq n+1$ for every subset C of $X-A$ which is closed in X; for metric X, this of course implies that dim $(X-A) \leqq n+1$). (Received April 6, 1955.)

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