BOOK REVIEWS

Analytische Fortsetzung. By L. Bieberbach. (Ergebnisse der Mathematik und ihrer Grenzgebiete, N.S., no. 3.) Berlin, Springer, 1955. 4+168 pp. 24.80 D.M.

The general scope of this monograph is approximately the same as the Chapter of the same title in the author's *Lehrbuch der Funktionentheorie* (Chap. 7, vol. II, Teubner, 1931) brought up to date by the developments of the last 25 years. Much of this was initiated by Pólya and those who have been influenced by him. The central problem can be stated somewhat as follows: how are properties of the coefficient sequence $\{a_n\}$ reflected in the behavior of $\sum a_n z^n$, and conversely? Three chief tools form the basis for the development, and are discussed in Chapter 1; these are the method of associated functions, linked with the Laplace-Borel transform, the Hadamard multiplication theorem, and the Euler series transformation. If $A(z) = \sum_0^\infty c_n z^n/n!$ is an entire function of exponential type, its Borel transform is $a(z) = \sum_0^\infty c_n/z^{n+1}$. The rate of growth of A along rays is closely connected with the region of regularity of $a(z)$; in turn, one may immediately obtain information about the location of singularities of the associated function $g(z) = \sum_0^\infty A(n)z^n$. The classical multiplication theorem of Hadamard deals with the three power series, $a(z) = \sum a_n z^n$, $b(z) = \sum b_n z^n$, and $c(z) = \sum a_n b_n z^n$. An open set S is a star (at 0) if $\lambda S < S$ for all λ, $0 \leqq \lambda \leqq 1$. If A and B are star sets, their star-product is $A \odot B = (A' \cdot B')'$. If $a(z)$ is regular for $z \in A$, and $b(z)$ for $z \in B$, then $c(z)$ is regular at least in $A \odot B$. It is not true [as other books have not always pointed out] that the singularities of $c(z)$ necessarily have the form $\alpha\beta$ where α and β are, respectively, singularities of $a(z)$ and $b(z)$. However, every suitably restricted boundary point of $A \odot B$ which is a singularity for $c(z)$ has this form. The Euler transformation provides a simple necessary and sufficient condition that a power series $\sum a_n z^n$ have 1 as a singularity.

It is difficult to give more than an indication of the skill and elegance with which the author combines these simple and familiar tools to obtain a unified treatment of a selected portion of the theory of Taylor series. In Chapters 2 and 3, one finds a discussion of the effect of coefficient gaps upon the presence and location of singularities of a power series, starting with the general theorems of Fabry and Pólya, and ending with more detailed and specialized theorems of Wilson, MacIntyre, Mandelbrojt and many others. In particular, §2.1 is devoted to a clarification [with enlightening comments] of Fabry's original proof. In Chapter 4, the subject of study is the class of power series with unit radius of convergence, and various justifications of the general remark that most of these have the unit circle for a cut; for example [Boerner], the set of $\beta=(\beta_0, \beta_1, \cdots)$ with $|\beta_j| \leqq \pi$ for which $\sum a_n \exp (i\beta_n)z^n$ is continuable has measure zero, while [Pólya-Hausdorff] the class of non-continuable power series is open and dense in the space of power series.

Another general remark of somewhat similar nature is that whenever $\{a_n\}$ is a sufficiently "nice" sequence, $\sum a_n z^n$ is either rational, or has the circle of convergence for a cut. This is illustrated by the material of Chapter 6 which revolves around the classical theorems of Eisenstein, Szegö, Pólya, and Carlson. As an instance of the method of associated functions, the author digresses in §6.3 to include a sketch of some known results dealing with integral valued entire functions. If $A(z)$ is an entire function of exponential type such that $A(n) \in D$ for $n=0, 1, \cdots$ where D is a domain of algebraic integers, then one may consider the associated function $g(z) = \sum_0^\infty A(n)z^n$. If A is of sufficiently slow growth, $g(z)$ is regular in a set of mapping radius greater than 1; if D is either the rational integers, or a quadratic complex domain, then the Pólya-Carlson theorem may be applied to show that $g(z)$ is a rational function, from which the form of $A(z)$ may be obtained. Finally, Chapter 5 deals with certain consequences of the Hadamard multiplication theorem, and Chapter 7 discusses the connection between a power series $\sum a_n z^n$ and the related series $\sum \phi(a_n)z^n$ where $\phi(z)$ is a preassigned analytic function.

The author has not entered upon the general coefficient problems for schlicht functions and for bounded functions, nor has he discussed analytic continuation of power series by means of summability. Within his chosen framework, he has produced a remarkably interesting and coherent summary of recent work and literature.

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Theory of differential equations. By E. A. Coddington and N. Levinson. New York, McGraw-Hill, 1955. 14+429 pp. $8.50.

It has become fashionable of late, in various mathematical centers, to present the fundamental tools of analysis, real and complex variable theory, in an increasingly abstract manner to those most defenseless, namely fledgling graduate students. In the process, motivation for the introduction of new concepts has been on the whole by-passed as an atrophied relic of those early pioneer days when mathematicians were forced to consort with astronomers and physicists, and indeed,