BOOK REVIEWS

Linear inequalities and related systems. Ed. by H. W. Kuhn and A. W. Tucker. Annals of Mathematics Studies, no. 38, Princeton, Princeton University Press, 1956. 22+322 pp. $5.00.

This is the first collection of papers devoted to a subject standing midway between the study of convex sets in linear spaces which it uses and that of certain optimization problems, including minimax approximation, linear programming and matrix games, to which it is applied, and which in turn bear on decision processes in statistical and economic theory. Delineated by Fourier in connection with mechanical equilibrium as well as minimax error problems, given (at the occasion of determining a fundamental region for the set of all positive definite quadratic forms under a discrete transformation group) its substance and essential results by Minkowski, developed by Farkas, Dines, Schoenberg, Weyl and others and systematically established in the reviewer's inaugural dissertation, the theory of linear inequalities with its ramifications has, in the last decade, attracted increasing attention in this country, and also in the U.S.S.R., while modern high-speed computing has made its practical application feasible.

The present study contains nine papers dealing with general linear systems and their geometric and extremizational counterparts, followed by nine papers on special cases. Of the latter, those by Dantzig, Fulkerson, Heller, Hoffman, Kruskal, Kuhn, and Tompkins consider systems whose basic solutions belong to the ring (not merely field) of coefficients, in particular array problems; the paper by Gale shows that, in $2n$-space, convex polyhedra with an arbitrary number of vertices exist every n of which form a face (cf. Motzkin, *Comonotone curves and polyhedra*, Bull. Amer. Math. Soc., vol. 63 (1957), p. 35); Kuhn and Gale re-prove the theorems of Wald and von Neumann on economic equilibrium; a related economic problem is brought into game-theoretic form by Thompson and shown to have only a finite number of solutions.

Of the papers on general systems, Fan's is the first systematic study of linear inequalities which includes infinite systems, systems in infinite dimensional linear spaces and also systems in complex linear spaces. Duffin, *Infinite programs*, likewise deals with linear operators. A new algorithm for linear programs is given by Dantzig, Ford and Fulkerson. Wolfe considers matrix games with polyhedral, instead of simplicial, strategy sets. Mills establishes an elegant theorem on the variation of the value of a game as function of its matrix.

The first four papers, by Goldman and Tucker, culminate in a *Theory of linear programming*, preceded (also logically) by an examination of the structure of polyhedral convex cones under set inclusion, by variants of the reviewer's theorems on the resolution of a polyhedral convex set into the sum of a polyhedron and a polyhedral cone and on systems related by matrix transposition, and by Tucker's opening paper on *Dual systems of homogeneous linear relations*, which consolidates previous transposition theorems and introduces the concept of slackness, stressing the importance of equated constraints (casually appearing in Stiemke, Math. Ann. vol. 76, p. 342; cf. also Miller, Amer. Math. Monthly vol. 17, p. 137; Černikov, Mat. Sb. N.S. vol. 38 (80), pp. 506–507).

As might be expected from the editors of Annals Studies 24 and 28, the value of this representative cross-section of recent research in the field is enhanced by careful revision, pleasant appearance, a clear and instructive résumé of the individual papers in the preface to the study, and an extensive and up-to-date bibliography.

THEODORE S. MOTZKIN

Einführung in die Geometrie der Waben. By W. Blaschke. (Elemente der Mathematik vom höheren Standpunkt aus, Band IV) Birkhäuser Verlag, Basel and Stuttgart, 1955. 108 pp. SFr 15.25.

This monograph contains four chapters. Chapter I deals with 3-honeycombs $\mathfrak{W}$ of curves in a Euclidean plane E_2, called 3-webs of curves in *Geometrie der Gewebe* (Springer, Berlin, 1938) by the author and G. Bol. For a $\mathfrak{W}$ three Pfaffian forms are introduced, and the connection γ and the curvature k are derived by exterior differentiation. It is proved that a necessary and sufficient condition for $\mathfrak{W}$ to be a hexagonal honeycomb is that k vanishes identically. Furthermore k is expressed in terms of a honeycomb function W. There are also obtained a complete system of invariants and a canonical expansion of the function W, from which G. Thomsen's geometric interpretation of the curvature k is deduced. Rotations and conformal mappings of honeycombs are studied by means of complex Pfaffian forms.

Chapter II is devoted to 4-honeycombs $\mathfrak{W}$ of surfaces in a three-dimensional Euclidean space E_3. As in Chapter I for a $\mathfrak{W}$ the author introduces four Pfaffian forms and obtains the connection γ, the curvatures a_1, a_2, a_3, a complete system of invariants and a canonical expansion of a honeycomb function W. It is proved that a necessary and sufficient condition for a $\mathfrak{W}$ to be an octahedral honeycomb is