TOTALLY ORDERED COMMUTATIVE SEMIGROUPS

A. H. CLIFFORD[1]

Let $S(+, <)$ be a system consisting of a set S endowed with an associative binary operation $+$ and a total (=linear =simple) order relation $<$. The composition $+$ and the relation $<$ may be connected by either or both of the following conditions.

MC (*Monotone Condition*). If a and b are elements of S such that $a<b$ then $a+c\leqq b+c$ and $c+a\leqq c+b$ for all c in S.

CC (*Continuity Condition*). $(x, y)\to x+y$ is a continuous mapping of $S\times S$ into S, where S is endowed with the order topology.[2]

We shall call S an *ordered semigroup* (abbreviated "o.s.") if MC holds, and an *ordered topological semigroup* (abbreviated "o.t.s.") if CC holds. §2 below (Theorems 1–6) deals with the former, and §3 (Theorems 7–10) with the latter. An o.t.s. is an instance of a mob in the sense of A. D. Wallace [**30**].

If an o.s. S is a group with respect to $+$, then S is an ordered group, as customarily defined. In this case CC also holds. On the other hand, in each of Theorems 7–10, it turns out that MC emerges as a consequence of CC and other hypotheses. In general, however, MC and CC are independent.

An o.s. S satisfies the *strict* MC, i.e. $a<b$ implies $a+c<b+c$ and $c+a<c+b$, if and only if it is *cancellative*, i.e. $a+c=b+c$ or $c+a=c+b$ implies $a=b$.

In spite of the title, we shall not assume that S is *commutative*, i.e. $a+b=b+a$ for all a, b in S. In each of Theorems 7–10 and also Theorem 1 (Hölder 1901), commutativity will not be a hypothesis, but will be a conclusion of the theorem.

The bibliography (25 items) lists all papers known to me dealing with o.s.'s or o.t.s.'s which are not necessarily ordered groups. (Although every group is of course also a semigroup, the "theory of semigroups" does not presume to include the vastly larger theory of groups.) Items [**9**; **10**; **16**]; and [**17**] contain results on o.t.s.'s which

An address delivered before the Coral Gables meeting of the Society on November 30, 1957 by invitation of the Committee to Select Hour Speakers for Southeastern Sectional Meetings; received by the editors March 3, 1958.

[1] This paper was prepared with the partial support of the National Science Foundation grant to Tulane University.

[2] For order topology, see for example Garrett Birkhoff [**26**, pp. 39–41]. Numbers in brackets refer to the bibliography and general references listed at the end of the paper.

are subsidiary to the main purpose of the paper. Items [18] and [19] are principally concerned with partially ordered semigroups, but contain much of interest for totally ordered ones.[3] In all the rest, either o.s.'s or o.t.s.'s are the chief concern. It would be clearly impossible to give an adequate account of all of these in an hour. I have chosen to present ten theorems that appeal to me as interesting and significant.

The general references, items [26–30], contain pertinent material, but nothing specifically on the subject of o.s.'s or o.t.s.'s.

NOTATION. Let A and B be subsets of an o.s. or o.t.s. Then: (1) $A+B$ means the set of all $a+b$ with a in A, b in B; (2) $A<B$ means $a<b$ for all a in A, b in B; (3) $A\backslash B$ means the set of all elements in A but not in B.

The whole paper has been expressed in additive notation for the sake of uniformity. Most of the references use multiplicative notation, but additive was chosen because it seems more natural for the basic examples ("fundamental semigroups") given in §1. These are denoted by P, P_1, P_1^*, Z, and Z_n (n any positive integer). Other fixed symbols are $P[1]$, $P(1)$, $Z[n]$, defined in §1, and R for the additive ordered group of all real numbers.

ABBREVIATIONS.

o.s. = ordered semigroup.

o.c.s. = ordered commutative semigroup.

o.t.s. = ordered topological semigroup.

n.o. = naturally ordered.

pos.o. = positively ordered.

(p.o. = partially ordered).

1. Basic definitions and examples.

Let S be a semigroup. An element 0 of S is called an *identity element* if $0+a=a+0=a$ for all a in S; it is unique if it exists. An element ∞ of S is called an *absorbent element* if $\infty+a=a+\infty=\infty$ for all a in S; it is also unique if it exists. An element e of S is *idempotent* if $e+e=e$.

Let $a\in S$, and let n be a positive integer. By na we mean $a+a+\cdots+a$ (n terms), and we call na a *natural multiple* of a. The set of natural multiples of a is a subsemigroup of S called the *cyclic subsemigroup of S generated by a*. The number of distinct multiples of a is the *order* of a.

If S is an o.s., and a has finite order n, then either

$$a < 2a < 3a < \cdots < (n-1)a < na = (n+1)a = (n+2)a = \cdots$$

[3] The same is true of Item [25a], of which I was unaware at the time this paper was written.

or the dual thereof; in either case, na is idempotent. By the *dual* of a statement we mean that obtained from it by interchanging $<$ and $>$. By the *dual* of an o.s. or o.t.s. $S(+, <)$, we mean $S(+, >)$.

Let S be a semigroup. A subset J of S is called an *ideal* if $J+S\subseteq J$ and $S+J\subseteq J$. By the *Rees difference-semigroup* $S-J$ (Rees [**28**, p. 389]) we mean the semigroup $(S\setminus J)\cup\{\infty\}$, where ∞ does not represent any element of S, with addition $\dot{+}$ defined as follows (a, b in $S\setminus J$):

$$a \dot{+} b = \begin{cases} a+b & \text{if } a+b \notin J, \\ \infty & \text{if } a+b \in J; \end{cases}$$

$$\infty \dot{+} a = a \dot{+} \infty = \infty \dot{+} \infty = \infty.$$

Let S be an o.s. or o.t.s. A subset A of S is called *convex* if $a\in A$, $b\in A$, $x\in S$, and $a<x<b$ imply $x\in A$. If J is a convex ideal, then $S-J$ can be ordered by retaining the original order in $S\setminus J$ and declaring, for any a in $S\setminus J$, $a<\infty$ or $a>\infty$ according to whether $a<J$ or $a>J$. If S is an o.s., so is $S-J$. If S is an o.t.s., and J is a closed interval, then $S-J$ is also an o.t.s. We regard the sets $\{x\mid x\in S, x\leqq c\}$ and $\{x\mid x\in S, x\geqq c\}$, with c a fixed element of S, as being closed intervals.

Let S and S' be two o.s.'s or two o.t.s.'s. A one-to-one mapping f of S into S' is called an *isomorphism* if $f(a+b)=f(a)+f(b)$ and if $a<b$ implies $f(a)<f(b)$, for all a and b in S. We then say that S is *embedded* in S'. If f maps S *onto* S', then we say that S and S' are isomorphic.

Let P be the ordered additive semigroup of all positive real numbers, and Z that of all positive integers. Let

$$P[1] = \{x \mid x \in P, x \geqq 1\}, \qquad P(1) = \{x \mid x \in P, x > 1\},$$
$$Z[n] = \{m \mid m \in Z, m \geqq n\}, \qquad n \text{ a fixed positive integer.}$$

$P[1]$ and $P(1)$ are convex ideals in P, likewise $Z[n]$ in Z, and we may form the ordered Rees difference-semigroups

$$P_1 = P - P[1], \qquad P_1^* = P - P(1), \qquad Z_n = Z - Z[n].$$

We may visualize P_1 as the half-closed real interval $(0, 1]$ with addition $\dot{+}$ defined by $a\dot{+}b=\min\{a+b, 1\}$. We may visualize P_1^* as $(0, 1]\cup\{\infty\}$ with

$$a \dot{+} b = \begin{cases} a+b & \text{if } a+b \leqq 1, \\ \infty & \text{if } a+b > 1. \end{cases}$$

Since $P[1]$ is closed, P_1 is continuous (as well as montone); P_1^* is monotone but not continuous. Every infinite cyclic o.s. is isomorphic with Z or its dual; every finite cyclic o.s. of order n is isomorphic with Z_n or its dual. Being discrete, Z and Z_n are continuous.

P, P_1, P_1^*, Z, and Z_n (n a positive integer) will be called *the fundamental semigroups*. Theorems 5, 6, and 10 below show that extensive classes of o.s.'s and o.t.s.'s are constructible from the fundamental semigroups. Theorems 1(A), 4(A), 8, and 9 characterize individual fundamental semigroups, or slight modifications thereof.

Let S be an o.s. We call S *positively ordered* (abbreviated "pos.o.") if $a+b\geqq a$ and $a+b\geqq b$ for all a, b in S. This differs from Yamada's definition, [**25**, p. 17], which requires that $a+b>a$ and $a+b>b$.

If S contains an identity element 0 as its lower endpoint, then it is positively ordered; for, using the MC, $a\geqq 0$ implies $a+b\geqq b$, and $b\geqq 0$ implies $a+b\geqq a$. Conversely, let S be positively ordered. If S contains an identity element 0, then 0 is the lower endpoint of S; and if S does not contain an identity element, one can be adjoined to S at its lower end.

We say that an o.s. S is *naturally ordered* (abbreviated "n.o.") if it is positively ordered, and $a<b$ implies that $a+x=y+a=b$ for some x and y in S. An o.c.s. (commutative o.s.) S is n.o. if and only if the following is true: $a\leqq b$ (a, b in S) if and only if $a=b$ or $a+x=b$ for some x in S. The foregoing may be taken as the definition of a binary relation $\leqq$ in any commutative semigroup S. This relation is reflexive, transitive, and montone; it is indeed just the usual division relation when S is written multiplicatively. It is a total ordering of S if and only if the trichotomy condition is satisfied: for any a, b of S, exactly one of the relations $a<b$, $a=b$, $b<a$ holds. Klein-Barmen [**14**; **15**] calls an n.o.c.s. with identity element a *linear holoid*. If S is n.o., then every subsemigroup of S is pos.o. But there exist pos.o.c.s.'s which cannot be embedded in an n.o.c.s., e.g. Nakada's Example 10, [**19**, p. 83]. The fundamental semigroups are all n.o. On the other hand, $P[1]$, $P(1)$, and $Z[n]$ for $n>1$, are pos.o. but not n.o.

An o.s. S will be called *archimedean* if the following condition holds. Let a and b be any elements of S neither of which is the identity element of S (if such exists). (1) If $2a\geqq a$ and $2b\geqq b$, then there exists a positive integer n such that $na\geqq b$; and (2) if $2a\leqq a$ and $2b\leqq b$ then $na\leqq b$ for some n. If S is pos.o., (2) can be omitted and the hypothesis in (1) is redundant.

An ordered set S is called (*conditionally*) *complete* if every subset of S bounded from above has a least upper bound. This is equivalent to the dual statement. I shall omit the modifier "conditionally."

Every complete ordered abelian group is archimedean, but this is not so for ordered commutative semigroups in general. The fundamental semigroups are all complete and archimedean.

A semigroup S will be called *nil* if it contains an absorbent element ∞, and if every element a of S is *nilpotent*: $na=\infty$ for some positive integer n. P_1, P_1^*, and Z_n are nil. If a pos.o.c.s. S is nil, then ∞ is the greatest element of S, and S is clearly archimedean.

2. **Algebraic theory.** In this section we deal exclusively with o.s.'s S, and in fact with commutative ones except for Theorem 1.

Theorem 1, due to Hölder [**11**], is the earliest and most fundamental in this subject. Its statement has been rephrased in accordance with our present terminology. Immediate and celebrated consequences of Theorem 1 are: (A′) every complete ordered group is isomorphic with the additive group R of all real numbers; (B′) every archimedean ordered group can be embedded in R, and in particular must be abelian.

THEOREM 1 (HÖLDER 1901). *Let S be a cancellative, naturally ordered semigroup without identity element and without a least element.* (A) *S is isomorphic with P if and only if it is complete.* (B) *S can be embedded in P if and only if it is archimedean.*

This should be supplemented by the following, first proved by Huntington [**12**, p. 271, Case I; **13**, Theorems I′ and II′]. I hasten to add that this is only a byproduct of these two papers, the main objective of which was to give independent and categorical systems of axioms for P, Z, and the additive ordered group of positive rationals.

SUPPLEMENT (HUNTINGTON 1902). *Let S be a cancellative, naturally ordered semigroup without identity element and having a least element. If S is archimedean, then it is complete, and is isomorphic with Z.*

If G is an ordered abelian group, let $G_+=\{x \mid x\in G,\ x\geqq 0\}$. Any subsemigroup of G is a cancellative o.c.s. Conversely, we have the following theorem, the first part of which is essentially well-known. The embedding of Z in the ordered additive group of all integers is a familiar special case. It does not, however, seem to be in the literature in general form prior to being given explicit expression independently by Dov Tamari [**21**] in France, Alimov [**2**] in Russia, and Nakada [**18**] in Japan. The last two assertions are made by Nakada in Theorems 5 and 7 of [**18**].

THEOREM 2 (TAMARI 1949, ALIMOV 1950, NAKADA 1951). *Every cancellative ordered commutative semigroup S can be embedded in an ordered abelian group G, unique to within isomorphism, such that every* *element of G is the difference of two elements of S. S is contained in the positive part G_+ of G if and only if it is positively ordered. $S=G_+$ or $G_+\backslash\{0\}$ if and only if S is naturally ordered.*

This theorem does not hold as it stands if we remove the hypothesis that S be commutative. In 1953, Chehata [**3**] and Vinogradov [**24**] independently gave the same example of an ordered cancellative semigroup which cannot be embedded in a group.

It must not be supposed that embeddability in an ordered abelian group places the theory of cancellative o.c.s.'s outside the scope of the present theory. (The theory of subsemigroups of a group belongs to the theory of semigroups as well as to group theory!) For example, Yamada [**25**] characterizes an interesting class of subsemigroups of P.

If S is a cancellative o.c.s., and G is its ordered difference-group, it is not always easy to predict properties of G from those of S. As an illustration, S may be archimedean but G nonarchimedean. For example, let S be the free commutative semigroup generated by two symbols x and y, i.e. S consists of all $mx+ny$ (m and n non-negative integers, not both zero), with $mx+ny=m'x+n'y$ if and only if $m=m'$ and $n=n'$. Define $mx+ny<m'x+n'y$ if $m+n<m'+n'$, or if $m+n=m'+n'$ and $m<m'$. Then it is easy to see that S is an archimedean, cancellative o.c.s. But G is not archimedean, for $x>0$ and $x-y>0$, but $n(x-y)<x$ for all n, since $nx<x+ny$ in S.

Alimov [**2**] gives an interesting criterion that G be archimedean. For simplicity, assume that S is a positively ordered, cancellative o.c.s. Two elements a, b of S are said to form an *anomalous pair* if, for every positive integer n, $na<nb<(n+1)a$. Then Alimov's criterion is: *G is archimedean if and only if S contains no anomalous pair.* The condition is plainly necessary, since, by Hölder's Theorem, if G is archimedean, it can be embedded in the additive group of all real numbers. To show the sufficiency, suppose G is not archimedean. Then there exist elements a, b, c of S such that $a>c$ and $n(a-c)<b$ for every n. Then $(b+c,\ b+a)$ is an anomalous pair. For $nc<na<b+nc$, whence

$$n(b+c) < n(b+a) < nb+b+nc < (n+1)(b+c).$$

Let I be an ordered set. To each i in I let correspond a pos.o.c.s. S_i. For $i\neq j$ in I, we assume that S_i and S_j are disjoint. Let $S=\bigcup_{i\in I} S_i$. Order S so that $S_i<S_j$ if $i<j$, and such that order within each S_i is the same as already defined. Define $+$ in S extending the given operation $+$ in each S_i, and such that if $a\in S_i$, $b\in S_j$, and $i<j$, then $a+b=b+a=b$. One easily verifies that S is also a pos.o.c.s. We call S the

ordinal sum of the ordered set $\{S_i | i \in I\}$ of pos.o.c.s.'s S_i. A pos.o.c.s. is called *ordinally irreducible* if it cannot be expressed as an ordinal sum of two or more subsemigroups.

THEOREM 3 (KLEIN-BARMEN 1942, IN PART). *Every positively (naturally) ordered commutative semigroup is uniquely expressible as an ordinal sum of an ordered set of ordinally irreducible positively (naturally) ordered commutative semigroups.*

This theorem, for n.o.c.s.'s, was found by Klein-Barmen [**14**] for the case in which S is finite or has the order type of the positive integers, and the general case was given in [**4**]. The proof for pos.o.c.s.'s is word-for-word the same as that given in [**4**] for n.o.c.s.'s, replacing "ideal" by "upper class," except for trifling changes in the proof of Lemma 1.1, p. 633. Logically, this should be the basic theorem, and that for n.o.c.s.'s derived therefrom by observing that a pos.o.c.s. is naturally ordered if and only if all of its ordinally irreducible components are naturally ordered. Remark 1 of [**4**, p. 643], gives the erroneous impression that one may define the ordinal sum of any ordered set of o.c.s.'s S_i ($i \in I$). If i is not the least element of I, then it is necessary that S_i be positively ordered; for if $j < i$, and $a \in S_j$, then $\{a\} \cup S_i$ is an o.c.s. with identity element at its lower end.

The following theorem is an amalgam of the Hölder-Huntington theorems and analogous results in [**4**] on noncancellative archimedean n.o.c.s.'s. The proof is not quite immediate, and will be given elsewhere [**6**].

THEOREM 4. *Let S be a naturally ordered commutative semigroup.* (A) *S is isomorphic with a fundamental semigroup if and only if it is complete and ordinally irreducible.* (B) *S can be embedded in a fundamental semigroup if and only if it is archimedean and has no identity element.*

The next theorem corrects an error in [**4**], namely the last statement in Remark 4, p. 644. The proof will be given in [**6**].

THEOREM 5. *Let S be a naturally ordered commutative semigroup, and let $S = \bigcup_{i \in I} S_i$ be its reduction into ordinally irreducible components S_i ($i \in I$). Then S is complete if and only if the following conditions are satisfied.*

(1) *The ordered set I is complete.*

(2) *For each i in I, S_i is isomorphic with a fundamental semigroup.*

(3) *If i is an element of I having no immediate successor, but is not the greatest element of I, then S_i must have a greatest element.*

(4) *If i is an element of I having no immediate predecessor, but is not the least element of I, then S_i must have a least element.*

(5) *If i, j is an adjacent pair of elements of I, with $i<j$, then either S_i must have a greatest or S_j a least element.*

We remark that the fundamental semigroups having a greatest element are P_1, P_1^*, and Z_n; those having a least element are Z and Z_p.

A natural two-fold objective is (1) to describe all complete o.c.s.'s, and (2) to describe all o.c.s.'s which can be completed, i.e. embedded in complete o.c.s.'s.[4] Theorem 5 solves (1) for the class of n.o.c.s.'s. In [6] it will be shown that an n.o.c.s. S can be embedded in a complete n.o.c.s. if and only if each ordinally irreducible component of S is archimedean.

In his book [29] on cardinal algebras, Tarski devotes a section (§13, pp. 175–189) to semigroups. By "semigroup," Tarski means what I would call a commutative, cancellative semigroup with identity. In Theorem 13.27, Tarski gives conditions on a semigroup S (in his sense) which are necessary and sufficient that S be a generalized cardinal algebra. The interesting cases are those in which the partial ordering in S is not total; for if it is total, S is then isomorphic with $\{0\}\cup P$ or $\{0\}\cup Z$. If we do not require cancellation, there are more interesting totally ordered cases. As the prime example, the cardinal algebra of all cardinal numbers is an n.o.c.s. which is the ordinal sum of Z and a well-ordered set of one-element semigroups. In the following theorem, we give necessary and sufficient conditions on an n.o.c.s. that it be a generalized cardinal algebra.

THEOREM 6. *Let S be a naturally ordered commutative semigroup, and let $S=\bigcup_{i\in I} S_i$ be its reduction into ordinally irreducible components S_i. Then S is a generalized cardinal algebra if and only if the following conditions hold.*

(1) *S has an identity element* 0.

(2) *Every countable subset of the ordered set I which is bounded from above has a least upper bound.*

(3) *For each i in I, S_i is isomorphic with P, Z, or Z_1.*

(4) *If S_i is isomorphic with P or Z, and i is not the greatest element of I, then i has an immediate successor j in I, and S_j is a one-element semigroup.*

(5) *If an element i of I is the least upper bound of a sequence of elements of I each less than i, then S_i is a one-element semigroup.*

If, in addition to satisfying these conditions, S has an absorbent ele-

[4] As a consequence of Krishnan's Theorem 1 [25a], every o.c.s. can be completed.

ment ∞, *then* S *is a cardinal algebra; if* S *does not have an absorbent element, then* $S \cup \{\infty\}$ *is a cardinal algebra.*

As an immediate corollary, we have the following result. Let S be an n.o.c.s. with identity element 0 and absorbent element ∞. Then S is a cardinal algebra if and only if it is an ordinal sum $S = \bigcup_{j \in J} S_j'$ of n.o.c.s.'s S_j' each isomorphic with $P \cup \{\infty\}$, $Z \cup \{\infty\}$, or $\{\infty\}$ such that the following condition is satisfied. Every strictly monotone increasing and bounded sequence of elements of J has a least upper bound in J, and for every element j of J which is the least upper bound of such a sequence, S_j is a one-element semigroup.

In a forthcoming paper [27], A. B. Clarke proves a reduction theorem for cardinal algebras analogous to Theorem 3 (namely, his Theorem 3.14). Also of interest in the present connection is his Theorem 4.7, roughly to the effect that any simple, archimedean cardinal algebra is isomorphic with one of the four algebras: $\{0\} \cup P \cup \{\infty\}$, $\{0\} \cup Z \cup \{\infty\}$, $\{0\} \cup \{\infty\}$, $\{0\}$.

3. Topological theory. Let S be an ordered set. We say that S is *bounded* if it has endpoints, i.e. greatest and least elements. We say that S is *dense* if, between any two distinct elements of S, there always lies a third element of S. Let S be endowed with the order topology. We then have two elementary theorems: (1) S is compact if and only if it is complete and bounded; (2) S is connected if and only if it is complete and dense.

By a *thread* we mean a connected o.t.s. By a *standard thread* we mean a bounded thread, one endpoint of which is the identity and the other the absorbent element of S.

In 1948, Aczél [1] showed that any cancellative, monotone thread S on a real interval is isomorphic with a subthread of the additive thread R of all real numbers. The following year, Dov Tamari [20] showed that the monotone condition is a consequence of the other assumptions, and also showed that every subthread of R is isomorphic with one of the following, or the dual thereof: R, P, $P \cup \{0\}$, $P[1]$, $P(1)$. It is, moreover, readily seen that the arguments used hold for general threads, not necessarily based on a real interval.

THEOREM 7 (ACZÉL 1948, TAMARI 1949). *Any cancellative thread is isomorphic with* R, P, $P \cup \{0\}$, $P[1]$, $P(1)$, *or with the dual of one of these.*

The systematic study of threads with idempotent endpoints was initiated by Faucett, [8] and [9]. The following is only one of many interesting results.

THEOREM 8 (FAUCETT 1955). *A standard thread with no interior idempotent element and no interior nilpotent element is isomorphic with* $\{0\} \cup P \cup \{\infty\}$ *or with the dual thereof.*

(We must warn the reader that we are maintaining the additive notation.)

The study was continued by Mostert and Shields [16], more or less incidentally to their work on semigroups on a manifold. The following two theorems were obtained for the case when S is based on a real interval, but it is readily seen that this restriction is not necessary. Theorem 10 is expressed in terminology quite different from that of its discoverers.

THEOREM 9 (MOSTERT AND SHIELDS 1957). *A standard thread with no interior idempotent element, but having at least one interior nilpotent element, is isomorphic with* $\{0\} \cup P_1$ *or with the dual thereof.*

THEOREM 10 (MOSTERT AND SHIELDS 1957). *Let I be any compact ordered set. If $i \in I$, and i has no immediate predecessor, let S_i be a one-element semigroup. If $i \in I$, and i has an immediate predecessor, let S_i be an isomorphic copy of either $P \cup \{\infty\}$ or P_1. Then the ordinal sum of the S_i ($i \in I$) is a standard thread, and conversely every standard thread has this structure.*

A proof of the converse can be based on Theorem 5 for Faucett showed that every standard thread S is naturally ordered (remark after Lemma 2 of [8]) and commutative (Lemma 5 of [8]). Since S is dense, each ordinally irreducible component S_i of S must be isomorphic with P, P_1, or Z_1. From (3) and (5) of Theorem 5, we see that every time P occurs as an S_i it is immediately followed by a Z_1, so we may merge these two into $P \cup \{\infty\}$. I shall omit the details.

A result closely related to Theorem 10 is given by Gleason, [10, Lemma 3]. In [17], Mostert and Shields determine the structure of a thread based on the open real line, $[0, \infty)$, with 0 and 1 playing their usual rôles (in multiplicative notation). A complete determination of all threads with idempotent endpoints is given in [5]. A complete determination of all bounded threads, one endpoint of which is the identity element, is given by Cohen and Wade [7].

BIBLIOGRAPHY

1. Jean Aczél, *Sur les opérations définies pour nombres réels*, Bull. Soc. Math. France, vol. 76 (1948) pp. 59–64.

2. N. G. Alimov, *On ordered semigroups*, Izv. Akad. Nauk SSSR. Ser. Mat. vol. 14 (1950) pp. 569–576 (Russian); Math. Rev. vol. 12, p. 480. Translated into English by R. P. Rich, Applied Physics Laboratory, Johns Hopkins University, Silver Spring, Maryland.

3. C. G. Chehata, *On an ordered semigroup*, J. London Math. Soc. vol. 28 (1953) pp. 353–356.

4. A. H. Clifford, *Naturally totally ordered commutative semigroups*, Amer. J. Math. vol. 76 (1954) pp. 631–646.

5. ———, *Connected ordered topological semigroups with idempotent endpoints* I, Trans. Amer. Math. Soc., vol. 88 (1958) pp. 80–98.

6. ———, *Completion of semi-continuous ordered commutative semigroups*, Duke Math. J., to appear.

7. Haskell Cohen and L. I. Wade, *Clans with zero on an interval*, Trans. Amer. Math. Soc. vol. 88 (1958) pp. 523–535.

8. W. M. Faucett, *Compact semigroups irreducibly connected between two idempotents*, Proc. Amer. Math. Soc. vol. 6 (1955) pp. 741–747.

9. ———, *Topological semigroups and continua with cut points*, Proc. Amer. Math. Soc. vol. 6 (1955) pp. 748–756.

10. Andrew M. Gleason, *Arcs in locally compact groups*, Proc. Nat. Acad. Sci. vol. 36 (1950) pp. 663–667.

11. O. Hölder, *Die Axiome der Quantität and die Lehre vom Mass*, Ber. uber d. Verh. d. K. Sächsischen Ges. d. Wiss. zu Leipzig, Math.-Phys. Cl. vol. 53 (1901) pp. 1–64.

12. E. V. Huntington, *A complete set of postulates for the theory of absolute continuous magnitude*, Trans. Amer. Math. Soc. vol. 3 (1902) pp. 264–279.

13. ———, *Complete sets of postulates for the theories of positive integral and positive rational numbers*, Trans. Amer. Math. Soc. vol. 3 (1902) pp. 280–284.

14. F. Klein-Barmen, *Über gewisse Halbverbände und kommutative Semigruppen* II, Math. Z. vol. 48 (1942–1943) pp. 715–734.

15. ———, *Ein Beitrag zur Theorie der linearen Holoide*, Math. Z. vol. 51 (1947–1949) pp. 355–366.

16. Paul S. Mostert and Allen L. Shields, *On the structure of semigroups on a compact manifold with boundary*, Ann. of Math. vol. 65 (1957) pp. 117–143.

17. ———, *On a class of semigroups on E_n*, Proc. Amer. Math. Soc. vol. 7 (1956) pp. 729–734.

18. O. Nakada, *Partially ordered abelian semigroups* I. *On the extension of the strong partial order defined on abelian semigroups*, J. Fac. Sci. Hokkaido Univ. vol. 11 (1951) pp. 181–189.

19. ———, *Partially ordered abelian semigroups* II. *On the strongness of the linear order defined on abelian semigroups*, J. Fac. Sci. Hokkaido Univ. vol. 12 (1952) pp. 73–86.

20. Dov Tamari, *Caractérisation des semi-groupes à un paramètre*, C. R. Acad. Sci. Paris vol. 228 (1949) pp. 1092–1094.

21. ———, *Groupoïdes reliés et demi-groupes ordonnés*, C. R. Acad. Sci. Paris vol. 228 (1949) pp. 1184–1186.

22. ———, *Groupoïdes ordonnés. L'ordre lexicographique pondéré*, C. R. Acad. Sci. Paris vol. 228 (1949) pp. 1909–1911.

23. ———, *Ordres pondérés. Caractérisation de l'ordre naturel comme l'ordre du semi-groupe multiplicatif des nombres naturels*, C. R. Acad. Sci. Paris vol. 229 (1949) pp. 98–100.

24. A. A. Vinogradov, *On the theory of ordered semigroups*, Ivanov. Gos. Ped. Inst. Uč. Zap. Fiz.-Mat. Nauki, vol. 4 (1953) pp. 19–21 (Russian); Math. Rev. vol. 17, p. 710.

25. M. Yamada, *Regularly totally ordered semigroups* I, Science Reports of Shimane University, 1957, pp. 14–23.

25a. V. S. Krishnan, *Les algèbres partiellement ordonnées et leurs extensions*, Bull. Soc. Math. France vol. 73 (1950) pp. 235–263.

General References

26. Garrett Birkhoff, *Lattice theory*, Amer. Math. Soc. Colloquium Publications, vol. 25, rev. ed., 1948.

27. A. B. Clarke, *On the representation of cardinal algebras by directed sums*, Trans. Amer. Math. Soc. (to appear).

28. D. Rees, *On semi-groups*, Proc. Cambridge Philos. Soc. vol. 36 (1940) pp. 387–400.

29. Alfred Tarski, *Cardinal algebras*, Oxford University Press, 1949.

30. A. D. Wallace, *The structure of topological semigroups*, Bull. Amer. Math. Soc. vol. 61 (1955) pp. 95–112.

The Tulane University of Louisiana