SOME ARITHMETIC PROPERTIES OF GENERALIZED BERNOULLI NUMBERS

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In a recent paper [2] Leopoldt has defined generalized Bernoulli numbers and polynomials in the following manner. Let f be a fixed integer $\geqq 1$ and $\chi(r)$ a primitive character (mod f). Put

$$\sum_{r=1}^{f} \chi(r) \frac{te^{(r+u)t}}{e^{ft}-1} = \sum_{n=0}^{\infty} B_\chi^n(u) \frac{t^n}{n!}, \qquad B_\chi^n = B_\chi^n(0).$$

For $f=1$, χ is the principal character and B_χ^n reduces to the ordinary Bernoulli number B_n. The main result of Leopoldt's paper is an analog of the Staudt-Clausen theorem.

In the present paper we obtain the following theorems, the first of which is a refinement of Leopoldt's analog of the Staudt-Clausen theorem. We assume $f>1$.

THEOREM 1. *If f is divisible by at least two different primes, then B_χ^n/n is an algebraic integer. If $f=p$, $p>2$, B_χ^n/n is an algebraic integer unless*

$$\mathfrak{p} = (p, 1-\chi(g)) \neq (1),$$

in which case

$$pB_\chi^n \equiv p - 1 \pmod{\mathfrak{p}^{n+1}};$$

if $f=p^\mu$, $p>2$, $\mu>1$, B_χ^n/n is integral unless

$$\mathfrak{P} = (p, 1-\chi(g)g^n) \neq (1),$$

in which case

$$(1-\chi(1+p)) \frac{B_\chi^n}{n} \equiv 1 \pmod{\mathfrak{P}};$$

g is a primitive root (mod p^r) *for all $r \geqq 1$. If $f=4$, then*

$$\frac{1}{n} B_\chi^n \equiv \begin{cases} 1/2 \pmod 1 & (n \text{ odd}), \\ 0 \pmod 1 & (n \text{ even}); \end{cases}$$

if $f=2^\mu$, $\mu>2$, then B_χ^n/n is integral.

THEOREM 2. *If $f=p^\mu$, then*

$$\sum_{s=0}^{r} (-1)^{r-s} \frac{B_\chi^{n+1+sw}}{n+1+sw} \equiv 0 \pmod{(q^n, q^{er})},$$

where q is a prime $\neq p$ and $q^{e-1}(q-1) \mid w$. If $f \neq p^\mu$, then (4.8) *holds for arbitrary primes q.*

THEOREM 3. *If p is a prime such that $p \nmid f$, $p^{e-1}(p-1) \mid m$, then*

$$\frac{1}{m+1} B_\chi^{m+1} \equiv \frac{1}{f}(1 - \chi(p)) \sum_{s=1}^{f} s\chi(s) \pmod{p^e}.$$

In particular, if $\chi(p) = 1$ or $\chi(-1) = 1$, then

$$\frac{1}{m+1} B^{m+1} \equiv 0 \pmod{p^e}.$$

In particular, for $f = 4$, Theorem 3 reduces to the following known result for the Euler numbers:

$$E_m \equiv \begin{cases} 0 \pmod{p^e}, & p \equiv 1 \pmod 4, \\ 2 \pmod{p^e}, & p \equiv 3 \pmod 4, \end{cases}$$

where $p^{e-1}(p-1) \mid m$.

The proof of these theorems makes use of various known properties of the ordinary Bernoulli numbers as well as the Eulerian numbers defined by [1]

$$\frac{1-\lambda}{e^t - \lambda} = \sum_{n=0}^{\infty} H_n(\lambda) \frac{t^n}{n!} \cdot$$

In particular we cite the representation

$$\frac{1}{n+1} B_\chi^{n+1} = \frac{\tau(\chi)}{f} \sum_{r=1}^{f} \frac{\bar{\chi}(r)\alpha^r}{1-\alpha^r} H_n(\alpha),$$

where

$$\tau(\chi) = \sum_{r=1}^{f} \chi(r)\alpha^r, \qquad \alpha = e^{2\pi i/f}.$$

REFERENCES

1. G. Frobenius, *Über die Bernoulli'schen Zahlen und die Euler'schen Polynome*, Preuss. Akad. Wiss. Sitzungsber. (1910) pp. 809–847.

2. H. W. Leopoldt, *Eine Verallgemeinerung der Bernoullischen Zahlen*, Abh. Math. Sem. Univ. Hamburg vol. 22 (1958) pp. 131–140.

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