AN ACTION OF A FINITE GROUP ON AN n-CELL WITHOUT STATIONARY POINTS

BY E. E. FLOYD AND R. W. RICHARDSON[1]

Communicated by Deane Montgomery, November 19, 1958

If G is a transformation group on a space X, then $x \in X$ is a stationary point if $gx = x$ for every $g \in G$. It has been an open problem, proposed by Smith [5] and by Montgomery [1, Problem 39], to determine whether every compact Lie group acting on a cell or on Euclidean space has a stationary point. Smith [4; 5] has shown the answer to be in the affirmative in case G is a toral group or a finite group of prime power order. In this note we give a simplicial action of A_5, the group of even permutations on five letters, on an n-cell without stationary points. Greever [3] has recently shown that the only finite groups of order less than 60 which could possibly act simplicially on a cell without stationary points are a certain class of groups of order 36.

We wish to thank P. E. Conner for his help and encouragement.

1. The coset space $SO(3)/I$. Let $SO(3)$ denote the group of all proper rotations of Euclidean 3-space E^3 and let $I \subset SO(3)$ be the group of rotational symmetries of the icosahedron. As a group, I is isomorphic to A_5 (see [9, pp. 16–18]) and hence is simple.

LEMMA 1. *The coset space* $SO(3)/I$ *has the integral homology groups of the* 3-*sphere* S^3.

PROOF. Let Q denote the algebra of quaternions and $Q_1 \subset Q$ the group of quaternions of norm one. Identify Q with E^4 and Q_1 with S^3. Let τ: $Q_1 \to SO(3)$ be the standard homomorphism, which is a two-to-one covering map. Set $I' = \tau^{-1}(I)$. Then τ induces a homeomorphism $Q_1/I' \approx SO(3)/I$.

The natural map π: $Q_1 \to Q_1/I'$ is a covering map and the group of covering translations is given by the action of I' on Q, by right multiplication. Since every covering translation preserves orientation it follows that Q_1/I' is an orientable 3-manifold and hence $H_3(Q_1/I') \approx H_3(SO(3)/I) \approx Z$ (here Z denotes the integers).

From covering space theory the fundamental group $\pi_1(Q_1/I')$ is isomorphic to I'. Thus $H_1(Q_1/I')$ is isomorphic to $I'/[I', I']$ where $[I', I']$ denotes the commutator subgroup of I'. Since I is simple,

[1] The first named author is the holder of a NSF Senior Postdoctoral Fellowship; the work of the second has been supported in part by contract A.F. 49(638)-104.

$[I, I]=I$. Also τ maps $[I', I']$ onto $[I, I]$; it follows that either $[I', I']=I'$ or $[I', I']\approx I$. But Q_1 contains only one element of order two. Since I contains fifteen elements of order two, $[I', I']$ is not isomorphic to I. Thus $I'=[I', I']$ and $H_1(Q_1/I')=0$. By Poincare duality it follows that $H_2(Q_1/I')=0$. The lemma follows.

2. **Action of I on $SO(3)/I$.** Let I act on $SO(3)/I$ by $g_1\cdot(gI)=g_1gI$. A point $\dot{g}=gI$ of $SO(3)/I$ is fixed under this action if and only if g belongs to the normalizer of I in $SO(3)$. But I is a maximal finite subgroup of $SO(3)$ (see [**9**, pp. 16–18]); furthermore, I is not included in any nonfinite proper closed subgroup of $SO(3)$, since this is not the case for the only two classes of such subgroups. Since I is not normal, it follows that I is its own normalizer. Hence there is exactly one stationary point of this action, and this is $\dot{e}$.

We say that the transformation group G acts *simplicially* on the space X if there exists a triangulation of X with respect to which the homeomorphism $g: X\rightarrow X$ is simplicial for every $g\in G$.

LEMMA 2. *The action of I on $SO(3)/I$ is simplicial.*

PROOF. Let $I'\times I'$ act on $Q(=E^4)$ by the rule $(q_1, q_2)\cdot q=q_1qq_2^{-1}$. This represents $I'\times I'$ as a finite group of orthogonal transformations of E^4. Hence we may find a triangulation of $S^3(=Q_1)$ such that the action of $I'\times I'$ is simplicial. The method is similar to one used by Whitney [**8**, p. 358, Lemma 3b]; we omit the details.

Now $e\times I'$ acts simplicially on Q_1, and the orbit space is Q_1/I'. By taking a barycentric subdivision, the triangulation of Q_1 induces a triangulation of the orbit space Q_1/I'. The action of $I'\times e$ on Q_1 induces an action of $I'\times e$ on Q_1/I' and since $I'\times e$ acts simplicially on Q_1 the induced action is simplicial with respect to the induced triangulation of Q_1/I'.

In the action of $I'\times e(=I')$ on Q_1/I' the effective group is $I'/\text{kernel}\,\tau$. Furthermore the homeomorphism τ_1 of Q_1/I' on $SO(3)/I$ is equivariant with respect to the action of $I'/\text{kernel}\,\tau$ on Q_1/I' and the action of I on $SO(3)/I$. It follows that the action of I on $SO(3)$ is simplicial.

3. **Action of I on a cell.** We may assume that the triangulation of Q_1 is C^1 in the sense of [**6**] and that e is a vertex. Since

$$\tau_1\cdot\pi: Q_1\rightarrow SO(3)/I$$

is a C^1-map the induced triangulation of $SO(3)/I$ is a C^1 triangulation. It follows that the closed star of the point I of $SO(3)/I$ is a 3-cell (see [**6**, p. 818, Theorem 5]). Let K denote the complex resulting if we remove the open star of the point I from $SO(3)/I$, and let $|K|$ denote the corresponding space. Then $|K|$ is acyclic (i.e. $H_i(|K|)=0$ for $i>0$, and $H_0(|K|)\approx Z$), and I acts simplicially on $|K|$ without stationary points.

Consider now the join $L=K\circ I$ of the complex K and the complex I, where I is the complex consisting of 60 vertices (the points of I) and no simplices of higher dimension. Since I acts on K, and I acts on I (by left multiplication), then I acts simplicially on L. In fact, $g\in I$ maps a line segment from $x\in K$ to $h\in I$ linearly into the line segment from $g(x)$ to gh. Furthermore, there are no stationary points on L. The polyhedron $|L|$ is a union of 60 cones over $|K|$, each pair intersecting in $|K|$. It follows that $|L|$ is acyclic, and also simply connected.

Let $(v_1, \cdots, v_n)$ denote the set of vertices of L. Each $g\in I$ induces a permutation η_g of the vertices of L; η_g may be considered as an element of the full symmetric group S_n on n letters.

Let $e_1, \cdots, e_n$ be basis vectors for E^n. Each element n of S_n determines a permutation of $(e_1, \cdots, e_n)$. If we extend linearly, n defines a linear transformation of E^n. This defines an action of S_n as a group of linear transformations of E^n.

Triangulate E^n so that the action of S_n is simplicial, and so that the simplex spanned by $e_1, \cdots, e_n$ is a simplex of the triangulation. Define an embedding f of L in E^n by setting $f(v_i)=e_i$ and extending f linearly to each simplex. Then f is equivariant. Hence I acts on $f(L)$, and without stationary points.

Let F_I be the set of points of E^n which are stationary under the action of I. Then $F_I\cap f(L)=\emptyset$. If we take sufficiently fine barycentric subdivisions we may assume that F_I does not intersect the first closed regular neighborhood of $f(L)$ (see [2, pp. 70–72 for definitions]), denoted by $N(f(L))$. Since I acts simplicially on E^n and $f(L)$ is invariant, it follows that $N(f(L))$ is also invariant. Since $f(L)$ is simply connected and acyclic, it follows from a theorem of J. H. C. Whitehead [7, Corollary 3, p. 298] that the regular neighborhood is a combinatorial n-cell. Thus I acts simplicially on the combinatorial n-cell $N(f(L))$ without stationary points.

BIBLIOGRAPHY

1. S. Eilenberg, *On the problems of topology*, Ann. of Math. vol. 50 (1949) pp. 247–260.

2. S. Eilenberg and N. Steenrod, *Foundations of algebraic topology*, Princeton, 1952.

3. J. J. Greever, *Fixed points of finite transformation groups*, Dissertation, University of Virginia.

4. P. A. Smith, *Fixed point theorems for periodic transformations*, Amer. J. Math. vol. 63 (1941) pp. 1–8.

5. ———, *Stationary points of transformation groups*, Proc. Nat. Acad. Sci. U.S.A. vol. 28 (1942) pp. 293–297.

6. J. H. C. Whitehead, *On C^1 complexes*, Ann. of Math. vol. 41 (1940) pp. 809–824.

7. ———, *Simplicial spaces, nuclei, and m-groups*, Proc. London Math. Soc. vol 45 (1939) pp. 243–327.

8. H. Whitney, *Geometric integration theory*, Princeton, 1957.

9. H. Zassenhaus, *Theory of groups*, New York, 1949.

UNIVERSITY OF VIRGINIA AND
INSTITUTE FOR ADVANCED STUDY
UNIVERSITY OF MICHIGAN AND
PRINCETON UNIVERSITY