# SMALL ISOTOPIES IN EUCLIDEAN SPACES AND 3-MANIFOLDS[1]

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Communicated by Edwin Moise, October 6, 1959

1. **Introduction.** The general type of question considered here is: what homeomorphisms of a space or of a set are obtained by isotopic deformations of a space by a small amount. Although questions of this type have only recently been treated explicitly and for their own sake (e.g. [2; 3; 5; 6; 7]) they had been handled implicitly in work done by Alexander [1] and Kneser [4] some 35 years ago. In fact this paper owes much to the method of Alexander, rediscovered in a slightly different form.

2. **Definitions.** Let $M$ be a manifold with boundary having a metric $d$.[2] Denote by $\mathcal{H}(M)$ the set of all homeomorphisms of $M$ onto itself. Define a function $\rho$ of $\mathcal{H}\times\mathcal{H}$ into the extended real number system as follows: $\rho(f, g)=\sup_{x\in M} d(f(x), g(x))$. $f$ and $g$ are $\epsilon$-isotopic if there is an isotopy $H_t$, $t\in I$, so that $H_0=f$, $H_1=g$ and if $t_1, t_2\in I$ then $\rho(H_{t_1}, H_{t_2})\leqq\epsilon$.

3. **Results.**

THEOREM 1. *If $f$ and $g$ are in $\mathcal{H}(E^n)$ and $\rho(f, g)=\epsilon<\infty$ then $f$ and $g$ are $\epsilon$-isotopic.*

PROOF. By the right invariance of $\rho$ it follows that $\rho(f, g)=\rho(1, gf^{-1})$ and if 1 and $gf^{-1}$ are $\epsilon$-isotopic under $H_t$ then $f$ and $g$ are $\epsilon$-isotopic under $H_tf$. Hence it suffices to prove the theorem for $f=1$. In this case, using vector notation for points in $E^n$, let $H_t(x)=tg(x/t)$ for $0<t\leqq 1$ and let $H_0=1$. The continuity of $H_t(x)$ in $t$ and $x$ is clear for $t>0$ and assured for $t=0$ by $d(x, H_t(x))=td(x/t, g(x/t))\leqq t\epsilon$.

This generalizes to $E^n$ (and slightly strengthens) a recent result of Sanderson for $E^3$ [7]. Alexander's result follows immediately by restricting the isotopy $H_t$ to the unit ball in $E^n$.

Another direction generalization can take is:

THEOREM 2. *Let $M$ be an arbitrary 3-manifold with boundary having a*

[1] The author is indebted to Professor R. H. Bing for suggesting the problem answered by Theorem 1 for $n=3$, and for directing the thesis, partially summarized here, to which the solution of that problem inexorably led. This research was supported by a Gulf Research and Development Company Fellowship.

[2] All Euclidean spaces $E^n$ will be assumed to have the usual metric. For more general manifolds the metric will be specified as needed.

*triangulation* $\Sigma$. *Let* $d$ *b the barycentric metric determined by* $\Sigma$ *and let* $\rho$ *be as defined above. Given any* $\epsilon > 0$ *there is a* $\delta > 0$ *so that if* $f, g \in \mathcal{H}(M)$ *and* $\rho(f, g) < \delta$, *then* $f$ *and* $g$ *are* $\epsilon$-*isotopic.*

PROOF. Only a sketch will be given here since the proof is quite long.

Again it suffices to let $f = 1$. The proof is in four stages. If we restrict $g$ to be close to 1 then on each 3-simplex $T$ in $\Sigma$ we can replace $g$ by $g'$ where $g' \mid \text{Bd } T = 1$ and $g'$ agrees with $g$ except in a small neighborhood of Bd $T$. An Alexander-type isotopy on each $T$ takes $g'$ onto 1 moving no point far. The global isotopy has the effect of deforming $g$ to a homeomorphism $g_1$ which is 1 except in a small neighborhood of the 2-skeleton. Next using [6] $g_1$ is modified to get $g_1'$ which is the same as $g_1$ on cubes built over the 2-simplexes in $\Sigma$ and is different from $g_1$ only near the 1-skeleton of $\Sigma$. An isotopy is pieced together again which deforms $g_1$ to a homeomorphism $g_2$ which is 1 except in a small neighborhood of the 1-skeleton. Two more reductions, near the 1-skeleton and 0-skeleton respectively, which are described on disjoint cubes near the 1-simplexes and vertices respectively, take $g_2$ onto the identity.

COROLLARY 1. *If* $M$ *is a compact* 3-*manifold with boundary, then* $h$ *is isotopic to* 1 *if and only if* $h = h_k h_{k-1} \cdots h_2 h_1$ *where each* $h_i$ *is the identity outside a polyhedral* 3-*cell.*

COROLLARY 2. *If* $L$ *is a tame compact* 2-*manifold in any* 3-*manifold* $M$ *and* $\epsilon > 0$, *there is a* $\delta > 0$ *so that if* $h$ *is any homeomorphism of* $L$ *into* $M$ *moving no point more than* $\delta$ *and if* $h(L)$ *is tame, then there is an* $\epsilon$-*isotopy of* $M$ *taking* $h(L)$ *onto* $L$ *pointwise and moving no point outside an* $\epsilon$-*neighborhood of* $L$.

This makes use of and generalizes a result of Sanderson [6].

COROLLARY 3. *If* $M$ *is a* 3-*manifold having triangulation* $\Sigma$ *and* $\epsilon > 0$ *there is a* $\delta > 0$ *so that if* $h$ *is a homeomorphism of the* 2-*skeleton* $K$ *of* $\Sigma$ *into* $M$ *moving no point more than* $\delta$ *and such that* $h(K)$ *is tame, then there is an* $\epsilon$-*isotopy of* $M$ *taking* $h(K)$ *onto* $K$ pointwise.

QUESTION. In Corollary 3 can $K$ be replaced by a 2-complex having no local separating points?

The author has been informed that G. M. Fisher and M. E. Hamstrom separately have obtained Theorem 2 for $M$ a compact 3-manifold with boundary and that the former also obtained Corollary 1.

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