RESEARCH ANNOUNCEMENTS

The purpose of this department is to provide early announcement of significant new results, with some indications of proof. Although ordinarily a research announcement should be a brief summary of a paper to be published in full elsewhere, papers giving complete proofs of results of exceptional interest are also solicited.

BEURLING'S TEST FOR ABSOLUTE CONVERGENCE OF FOURIER SERIES

BY R. P. BOAS, JR.

Communicated November 25, 1959

We say, with Beurling [1], that f is a contraction of g if $|f(x)-f(y)| \leqq |g(x)-g(y)|$. Beurling established the following theorem.

THEOREM 1. *If f and g are continuous even functions, of period 2π, with Fourier cosine coefficients c_n and g_n, if f is a contraction of g, and if $|g_n| \leqq \gamma_n$, where $\gamma_n \downarrow 0$ and $\sum \gamma_n < \infty$; then $\sum |c_n| < \infty$.*

Since saying that f is a contraction of g is essentially the same as saying that f is a Lipschitzian function of g, theorems like Theorem 1 have gained in interest since the recent discovery [3] that in general only analytic functions operate on all absolutely convergent Fourier series with preservation of absolute convergence.

I shall give an elementary proof of a generalization of Theorem 1. Further investigations along these lines by M. Kinukawa, M. and S. Izumi, and the author are in progress.

THEOREM 2. *Theorem 1 remains true when the hypothesis that $\gamma_n \downarrow 0$ is replaced by*

$$\sum_{n=1}^{\infty} n^{-3/2} \left\{ \sum_{k=1}^{n} k^2 \gamma_k^2 \right\}^{1/2} + \sum_{n=1}^{\infty} n^{-1/2} \left\{ \sum_{k=n+1}^{\infty} \gamma_k^2 \right\}^{1/2} < \infty. \tag{1}$$

Beurling's theorem follows from Theorem 2 since when $\gamma_n \downarrow 0$, or even when $n^{-\lambda}\gamma_n \downarrow 0$ $(\lambda > 0)$, the series in (1) converge if $\sum \gamma_n$ converges. This lemma has been proved by Konyushkov [4]; it is a corollary of more general results that I discuss elsewhere [2] by a different method; a proof by a still different method, that yields exact constants in the inequalities involved, has been obtained by S. Łojasiewicz.

Condition (1) can also be satisfied when $\{\gamma_n\}$ is not required to satisfy a condition of monotonicity. If $\sum n^{1/2}\gamma_n < \infty$, the left-hand side of (1) does not exceed, by Jensen's inequality,

$$\sum_{n=1}^{\infty} n^{-3/2} \sum_{k\leq n} k\gamma_k + \sum_{n=1}^{\infty} n^{-1/2} \sum_{k>n} \gamma_k$$
$$= \sum_{k=1}^{\infty} k\gamma_k \sum_{n\geq k} n^{-3/2} + \sum_{k=1}^{\infty} \gamma_k \sum_{n<k} n^{-1/2}$$
$$\leq \sum_{k=1}^{\infty} \gamma_k O(k^{1/2}).$$

Hence the hypothesis $\gamma_n \downarrow 0$ in Theorem 1 can be replaced by $\sum n^{1/2}\gamma_n < \infty$. In other words, the cosine series of f converges absolutely if f is a contraction of g and g has a derivative of order 1/2 that has an absolutely convergent Fourier series.

PROOF OF THEOREM 2. What is actually used in Beurling's proof (and in this one) is the condition

$$(2) \qquad \int_0^{\pi} | f(x+\delta) - f(x) |^2 dx \leq \int_0^{\pi} | g(x+\delta) - g(x) |^2 dx,$$

which may be thought of as saying that f is an average contraction of g. Take $\delta = \pi/n$, where n is a positive integer. By Parseval's theorem we can write (2) in the form

$$(3) \quad \sum_{k=1}^{\infty} c_k^2 \sin^2 (k\pi/(2n)) \leq \sum_{k=1}^{\infty} g_k^2 \sin^2 (k\pi/(2n)) \leq \sum_{k=1}^{\infty} \gamma_k^2 \sin^2 (k\pi/(2n)).$$

Let $\phi_n = \sum_{k=1}^{n} k | c_k |$. Then

$$\phi_n \leq n^{1/2} \left\{ \sum_{k=1}^{n} k^2 c_k^2 \right\}^{1/2},$$

and by partial summation

$$\sum_{n=1}^{N} | c_n | = \sum_{n=1}^{N} n^{-1}(\phi_n - \phi_{n-1})$$
$$= \sum_{n=1}^{N-1} \phi_n(n^{-1} - (n+1)^{-1}) + \phi_N/N$$
$$\leq \sum_{n=1}^{N} \phi_n/n^2 + \phi_N/N$$
$$\leq \sum_{n=1}^{N} n^{-3/2} \left\{ \sum_{k=1}^{n} k^2 c_k^2 \right\}^{1/2} + N^{-1/2} \left\{ \sum_{k=1}^{N} k^2 c_k^2 \right\}^{1/2}$$
$$= S_1 + S_2,$$

say. From (3) we have, since $\sin x \geqq 2x/\pi$ for $0 \leqq x \leqq \pi/2$,

$$\sum_{k=1}^{n} k^2 c_k^2 \leqq n^2 \sum_{k=1}^{n} c_k^2 \sin^2 (k\pi/(2n)) \leqq n^2 \sum_{k=1}^{\infty} \gamma_k^2 \sin^2 (k\pi/(2n)),$$

and hence

$$S_1 \leqq \sum_{n=1}^{N} n^{-1/2} \left\{ \sum_{k=1}^{\infty} \gamma_k^2 \sin^2 (k\pi/(2n)) \right\}^{1/2}, \tag{4}$$

$$S_2 \leqq N^{1/2} \left\{ \sum_{k=1}^{\infty} \gamma_k^2 \sin^2 (k\pi/(2n)) \right\}^{1/2}. \tag{5}$$

We must now show that S_1 and S_2 are bounded as $N \to \infty$. For S_2, we have

$$\begin{aligned} S_2^2 &\leqq N \sum_{k=1}^{N} k^2 \gamma_k^2 \pi^2/(4N^2) + N \sum_{k=N+1}^{\infty} \gamma_k^2 \\ &\leqq \frac{1}{4} \pi^2 N^{-1} \sum_{k=1}^{N} k^2 \gamma_k^2 + N \sum_{k=N+1}^{\infty} \gamma_k^2 \\ &= T_1 + T_2. \end{aligned}$$

The second series in (1) has decreasing terms, which must therefore be $O(1/n)$; hence $T_2 = O(1)$.

Call the first series in (1) $\sum n^{-3/2} A_n$; here A_n increases. We have

$$\sum_{N}^{2N} n^{-3/2} A_n \geqq A_N \sum_{N}^{2N} n^{-3/2} \geqq C A_N N^{-1/2},$$

with an irrelevant constant C, and hence $A_N^2/N \to 0$. Thus, in particular, $T_1 = O(1)$. This disposes of S_2.

We write

$$\begin{aligned} S_1 &\leqq \sum_{n=1}^{N} n^{-1/2} \left\{ \sum_{k=1}^{n} + \sum_{k=n+1}^{\infty} \right\}^{1/2} \\ &\leqq \sum_{n=1}^{N} n^{-1/2} \left\{ \sum_{k=1}^{n} \gamma_k^2 \sin^2 (k\pi/(2n)) \right\}^{1/2} \\ &\quad + \sum_{n=1}^{N} n^{-1/2} \left\{ \sum_{k=n+1}^{\infty} \gamma_k^2 \sin^2 (k\pi/(2n)) \right\}^{1/2} \\ &\leqq (\pi/2) \sum_{n=1}^{N} n^{-3/2} \left\{ \sum_{k=1}^{n} k^2 \gamma_k^2 \right\}^{1/2} + \sum_{n=1}^{N} n^{-1/2} \left\{ \sum_{k=n+1}^{\infty} \gamma_k^2 \right\}^{1/2}. \end{aligned}$$

If we assume (1), the two sums on the right are bounded. This completes the proof.

REFERENCES

1. A. Beurling, *On the spectral synthesis of bounded functions*, Acta Math. vol. 81 (1949) pp. 225–238.

2. R. P. Boas, Jr., *Inequalities for monotonic series*, J. Math. Analysis Appl. To appear.

3. H. Helson, J.-P. Kahane, Y. Katznelson and W. Rudin, *The functions which operate on Fourier transforms*, Acta Math. vol. 102 (1959) pp. 135–157.

4. A. A. Konyushkov, *Nailuchshie priblizheniya trigonometricheskimi polinomami i koeffitsienty Fure*, Mat. Sb. vol. 44 (86) (1958) pp. 53–84.

NORTHWESTERN UNIVERSITY