HANKEL TRANSFORMS AND VARIATION DIMINISHING KERNELS

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If $\phi(x)$ is a continuous function on $(-\infty, \infty)$ then we denote by $V[\phi]$ the number of variations of sign of $\phi(x)$ on $(-\infty, \infty)$. A measurable function $G(x)$ on $(-\infty, \infty)$ such that

$$G(x) \geqq 0, \qquad \int_{-\infty}^{\infty} G(x)dx = 1,$$

and such that

$$V[G * \phi] \leqq V[\phi]$$

for every bounded continuous ϕ will be called a variation diminishing $*$-kernel. Here

$$G * \phi \cdot (x) = \int_{-\infty}^{\infty} G(x - y)\phi(y)dy. \tag{1}$$

I. J. Schoenberg has proved that if G is a variation diminishing $*$-kernel then

$$\int_{-\infty}^{\infty} G(x)e^{-ixt}dx$$

is of the form

$$\left[e^{ct^2+ibt} \prod_k \left(1 - \frac{it}{a_k}\right) e^{it/a_k}\right]^{-1} \tag{2}$$

where the a_k's are real and $\sum_k a_k^{-2}$ is finite, b is real, and c is real and non-negative. Conversely every function of the form (2) is the Fourier transform of a variation diminishing $*$-kernel. See [8] and [9].

In the present note we will sketch an analogous theory in which certain convolutions of functions on $(0, \infty)$, associated with Hankel transforms replace the convolution (1).

Let γ be fixed, $0 \leqq \gamma$. We define

$$T(x) = 2^{\gamma-1/2}\Gamma(\gamma + 1/2)x^{1/2-\gamma}J_{\gamma-1/2}(x),$$
$$\mu(x) = x^{2\gamma+1}/2^{\gamma+1/2}\Gamma(\gamma + 3/2).$$

Let L be the set of measurable functions $f(x)$ on $(0, \infty)$ for which $\int_0^\infty |f(x)| d\mu(x)$ is finite. For $f \in L$ we set

$$f^{\wedge}(t) = \int_0^\infty T(xt)f(x)d\mu(x). \tag{3}$$

$f^{\wedge}(t)$ is the Hankel transform (of index γ) of $f(x)$. Let

$$D(x, y, z) = \frac{2^{3\gamma-5/2}\Gamma(\gamma + 1/2)^2}{\Gamma(\gamma)\pi^{1/2}} (xyz)^{-2\gamma+1}A(x, y, z)^{2\gamma-2}$$

where $A(x, y, z)$ is the area of a triangle whose sides are x, y, z if there is such a triangle and otherwise is zero. If $f(x)$ and $g(x)$ are defined on $(0, \infty)$ then we formally set

$$f \# g\cdot(x) = \int_0^\infty \int_0^\infty f(y)g(z)D(x, y, z)d\mu(y)d\mu(z).$$

It can be verified that if $f, g \in L$ then $f \# g \in L$ and $(f \# g)^{\wedge} = f^{\wedge} \cdot g^{\wedge}$; that is, the Hankel transform (3) behaves in regard to the convolution # exactly as does the Fourier transform with regard to ordinary convolution $*$ on the real line. This convolution associated with the Hankel transform was discovered by Delsarte [3] and [4]. See also the papers of Bochner [1] and [2] and the author [6].

If $\psi(x)$ is a continuous function on $(0, \infty)$ let $V[\psi]$ denote the number of changes of sign of $\psi(x)$ on $(0, \infty)$. A measurable function $H(x)$ on $(0, \infty)$ such that

$$H(x) \geqq 0, \qquad \int_0^\infty H(x)d\mu(x) = 1,$$

and such that

$$V[H \# \psi] \leqq V[\psi]$$

for every continuous bounded function ψ on $(0, \infty)$ will be called a variation diminishing #-kernel. Our principal result is the following.

THEOREM. *If $H(x)$ is a variation diminishing #-kernel then $H^{\wedge}(t)$ is of the form*

$$\left[e^{ct^2}\prod_k\left(1 + \frac{t^2}{a_k^2}\right)\right]^{-1} \tag{4}$$

where the a_k's are real and $\sum_k a_k^{-2}$ is finite, and where c is non-negative. Conversely every function of the form (4) *is the Hankel transform of a variation diminishing #-kernel.*

The demonstration of this result follows very closely the pattern established by Schoenberg. It is to be noted that for $\gamma=0$, this theorem is contained in Schoenberg's theorem as a special case.

Many important integral transforms can be reduced to the form $f=G*\phi$ where G is a variation diminishing $*$-kernel. Such transforms have a very extensive theory which is the subject of a book by D. V. Widder and the author [7]. It is evident that a parallel development can be carried through for the transforms $g=H\#\psi$. See also in this connection the paper by Fox [5].

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