MORSE INEQUALITIES FOR A DYNAMICAL SYSTEM

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1. Introduction. We consider dynamical systems (X, M), where X is a C^∞ vector field on a C^∞ closed manifold M satisfying the following conditions.

(1) There are a finite number of singular points of X, say $\beta_1, \cdots, \beta_k$, each of simple type. This means that at each β_i, the matrix of first partial derivatives of X in local coordinates has eigenvalues with real part nonzero.

(2) There are a finite number of closed orbits (i.e., integral curves) of X, say $\beta_{k+1}, \cdots, \beta_m$, each of simple type. This means that no characteristic exponent (see, e.g., [2]) of β_i, $i>k$, has absolute value 1.

(3) The limit points of all the orbits of X as $t \to \pm \infty$ lie on the β_i. In other words, denote by ϕ_t the 1-parameter group of transformations generated by X (as we do throughout this paper). Let

$$\alpha(y) = \operatorname*{limit\,set}_{t\to-\infty} \phi_t(y), \qquad \omega(y) = \operatorname*{limit\,set}_{t\to\infty} \phi_t(y), \qquad y \in M.$$

Then for each y, $\alpha(y)$ and $\omega(y)$ are contained in the union of the β_i.

(4) The stable and unstable manifolds of the β_i (see §2 for the definition) have normal intersection with each other. More precisely for each i let W_i be the unstable manifold and W_i^* the stable manifold of β_i and for $x \in W_i$ (or W_i^*) let W_{ix} (or W_{ix}^*) be the tangent space of W_i (or W_i^*) at x. Then for each i, j if $x \in W_i \cap W_j^*$,

$$\dim W_i + \dim W_j^* - n = \dim (W_{ix} \cap W_{jx}^*).$$

See [5] for example for more details.

(5) If β_i is a closed orbit there is no $y \in M$ with $\alpha(y) = \omega(y) = \beta_i$.

First we remark that systems satisfying (1)–(5) may be very important because of the following possibilities.

(A) It seems at least plausible that systems satisfying (1)–(5) form an open dense set in the space (with the C^1 topology) of all vector fields on M.

(B) It seems likely that conditions (1)–(5) are necessary and sufficient for X to be structurally stable in the sense of Andronov and Pontrjagin [1]. See also [6].

(A) and (B) have been proved for the case M is a 2-disk, [3] and [9].

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We expect to have more to say about this subject at another time. It is true that conditions (1)–(5) are independent.

With (X, M) as above let $\sigma_i = \dim W_i$. Then if $i \leq k$, σ_i is the number of eigenvalues associated to β_i with real part positive. Let a_q be the number of β_i, $i \leq k$ with $\sigma_i = q$. If $i > k$, σ_i is one more than the number of characteristic exponents of β_i with absolute value greater than one. Let b_q be the number of β_i, $i > k$, with $\sigma_i = q$.

The main goal of this paper is to prove

THEOREM 1.1. *Let (X, M) be a system satisfying (1)–(5), K any field, R_q the rank of $H^q(M, K)$, and $M_q = a_q + b_q + b_{q+1}$. Then M_q and R_q satisfy the Morse relations*

$$M_0 \geqq R_0,$$
$$M_1 - M_0 \geqq R_1 - R_0,$$
$$M_2 - M_1 + M_0 \geqq R_2 - R_1 + R_0,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\sum_{k=0}^{n} (-1)^k M_k = (-1)^n \chi$$

where $\dim M = n$ *and χ is the Euler characteristic of M with respect to K.*

Theorem 1.1 contains the Theorem of Èl′sgol′c [4] which excludes closed orbits. It also contains Reeb's theorem [11] which excludes singular points. However, both Èl′sgol′c and Reeb made the highly restrictive assumption[2] that no orbit joined saddle points (i.e., β_i, $i \leq k$ with $\sigma_i \neq 0, n$) or saddle type closed orbits (i.e., β_i, $i > k$ with $\sigma_i \neq 1, n$).

Also it follows from the following theorem which we prove elsewhere that Theorem 1.1 includes the classical theorem of Morse [8] for a function f on M with nondegenerate critical points.

THEOREM 1.2. *If $X = \text{grad} f$, f a C^∞ function on M with nondegenerate critical points, then X can be C^1 approximated by a C^∞ field Y on M such that (Y, M) satisfies (1)–(5) with no closed orbits.*

2. **Construction of the stable and unstable manifolds.** 2.1. Suppose β is a singular point of simple type of the C^∞ system (X, M). Let k be the number of eigenvalues associated to β with real part positive. Then (e.g., [2, p. 330]) there is a k dimensional C^∞ submanifold W of M passing through β such that if $x \in W$ then $\alpha(x) = \beta$. If $k = 0$, let

[2] Reeb has asked me to note that his footnote 3, 2nd paragraph, of [11, p. 62] (that this assumption is unnecessary) is incorrect

$W=\beta$. Then W is tangent at β to the linear subspace of the tangent space M_β of M at β defined by these k eigenvalues [2, p. 333]. W is called the *unstable manifold* of X at β. Let R^k denote Euclidean k-space considered as a vector space. We will show that W is the image of a continuous 1-1 onto map $f: R^k \to W$, with $f(0)=\beta$, and f is C^∞ with Jacobian of rank k except at 0. Consider the new system X^* obtained by reversing the direction of each vector of X on M. Then β is a simple singularity of X^* and the above applies to yield the unstable $(n-k)$-dimensional manifold W^* of X^* at β. Call W^* the *stable manifold* of X at β. Note W and W^* have normal intersection at β.

2.2. Suppose β is a closed orbit of (X, M) of simple type. Let $k-1$ be the number of characteristic exponents of β with absolute value greater than one. Then ([7] or [13]) there is a k-dimensional C^∞ submanifold W of M passing through β such that if $x \in W$ then $\alpha(x)=\beta$. If $k=1$ let $W=\beta$. Also W is tangent at each point y of β to the linear subspace of M_y defined by these $k-1$ characteristic exponents and the tangent vector of β at y. Call W the *unstable manifold* of X at β. We will show there is a continuous 1-1 onto map $f: R^{k-1} \times S^1 \to W$, with $f(0\times S^1)=\beta$, and except along $0\times S^1$ is C^∞ with Jacobian of rank k. Similarly to 2.1, one defines the *stable manifold* W^* of X at β whose dimension in this case is $n-k+1$.

We now construct the map f of 2.1.

There exists[3] a differentiably imbedded $(k-1)$-sphere K in W, which is everywhere transversal to X. Let S_0 be the unit sphere of R^k and $h: S_0 \to K$ be a diffeomorphism. (A diffeomorphism is C^∞ homeomorphism with a differentiable inverse.) Let ψ_t be the 1-parameter group of transformations of R^k generated by the vector field $Y(x)=x$ on R^k. For $x \in R^k$, $x \neq 0$, let $t(x)$ be the unique t such that $x/\|x\| = \psi_{t(x)}(x) \in S_0$. Then let $f(0)=\beta$ and $f(x)=\phi_{-t(x)} h \psi_{t(x)}(x)$. It is easy to check that $f: R^k \to W$ thus defined has the desired properties.

To construct the map $f: R^{k-1}\times S^1 \to W$ of 2.2, first let Y be the vector field $(x, 1)$ on $R^{k-1}\times S^1$. Then if ψ_t is the 1-parameter group of transformations generated by Y we have $\psi_t(x, 0)=(xe^t, t \bmod 2\pi)$. Let $R^{k-1}=R^{k-1}\times 0 \subset R^{k-1}\times S^1$, and C be the unit ball in R^{k-1}, $\partial C = S_0$. Define $q: R^{k-1}\to R^{k-1}$ by $q(x)=xe^{2\pi}$ and let $q^i S_0 = S_i$ for each integer i.

Let Q be a surface of section (i.e., transversal to X, see [6]) locally about a point of β in W, diffeomorphic to a $(k-1)$-cell. Then [6] the orbits of X define a diffeomorphism $h: Q\to Q$ in a neighborhood of $\beta\cap Q$ leaving $\beta \cap Q$ fixed. There is[3] a closed k-cell B differentiably

[3] By Liaponov theory for example.

imbedded in Q, $\partial B = F_0$ such that $h^{-1}(F_0)$ is contained in the interior of B. Let $h^i(F_0) = F_i$, $i \leqq 0$.

Let f be an orientation preserving diffeomorphism of a neighborhood V_0 of S_0 in R^{k-1} into a neighborhood of F_0 in Q. Then extend f to a neighborhood of $\bigcup_{i \leq 0} S_i$ in R^{k-1} into a neighborhood of $\bigcup_{i \leq 0} F_i$ in Q by the formula

$$(2.3) \qquad f(x) = h^{-i} f q^i(x), \qquad x \in \text{nbd. } V_i \text{ of } S_i.$$

This makes sense for an appropriate choice of the V_i's. Now consider the closed region U in R^{k-1} bounded by S_0 and S_{-1}. We have defined f in a neighborhood of the boundary ∂U of U. After restricting f to a smaller neighborhood of ∂U, f can be extended to a diffeomorphism of all of U into the region of Q bounded by F_0 and F_{-1}. This fact follows from arguments which are now standard in differential topology. We won't include them here. Then as in 2.3 we can extend f to a map of all of C into B which is a diffeomorphism except at $f(0) = \beta \cap Q$.

Next define f on $P = \{\psi_t(x) \mid x \in C,\ t < 0\}$ by the following: Let $\tau(x, \theta)$ be the smallest positive number such that $\phi_{\tau(x,\theta)} f \psi_{-\theta}(x, \theta)$ has θ as its second coordinate in a fixed product structure $Q \times \beta, (x, \theta) \in P$. Then let $f(x, \theta) = \phi_{\tau(x,\theta)} f \psi_{-\theta}(x, \theta)$. Define f: $0 \times S^1 \to \beta$ by $f(0 \times \theta) = \theta$.

Consider now the surface of section $S_{-1} \times S^1 = A$ in $R^{k-1} \times S^1$ and its image under f. Restrict f to the closure of the bounded component K of A. Finally extend f to all of $R^{k-1} \times S^1$ as follows. For $y \in R^{k-1} \times S^1 - K$ let $t(y)$ be the unique t such that $\psi_{t(y)}(y) \in A$. Then let $f(y) = \phi_{-t(y)} f \psi_{t(y)}(y)$. After a change of parameter near A, f will have our desired properties.

3. **Implications of** (1)–(5). Assume throughout this section that (X, M) is given as in §1. If β_i is a singular point then f_i: $R^k \to W_i$ is as in 2.1. If β_i is a closed orbit then f_i: $R^{k-1} \times S^1 \to W_i$ is as in 2.2.

LEMMA 3.1. *If $x \in M$, $\alpha(x) = \beta_i$, $\omega(x) = \beta_j$, then* dim $W_i \geqq$ dim W_j *and equality can occur only if β_j is a closed orbit.*

PROOF. Clearly $x \in W_i \cap W_j^*$ and by (4) we have that dim W_i + dim $W_j^* - n \geqq 1$. But dim $W_j^* = n -$ dim W_j if β_j is a singular point and dim $W_j^* = n -$ dim $W_j + 1$ if β_j is a closed orbit. Then 3.1 follows.

See [12] for the following.

LEMMA 3.2. *Suppose $W_i \cap W_j^* \neq \emptyset$ and $x \in W_j$. Then there exists a cell neighborhood H of x in W_j such that given $\delta > 0$, there is a $y \in W_i$ with $d(x, y) < \delta$ and if* dim W_i = dim W_j, *there is a subcell K of W_i such that H and K are within δ in a C^1 metric.*

Define $\partial W_j = \{\lim_{k\to\infty} f_j(x_k) \mid x_k$ any sequence in R^k with no lps.$\}$. Then let $\partial^2 W_j = \partial(\partial W_j)$, etc. Note Cl $W_i = W_i \cup \partial W_i$.

LEMMA 3.3. *If $W_i \cap W_j^* \neq \emptyset$, $\partial W_i \supset W_j$.*

This follows from 3.2.

LEMMA 3.4. *Suppose* dim W_i = dim W_k = dim W_j. *If $W_i \cap W_k^* \neq \emptyset$ and $W_k \cap W_j^* \neq \emptyset$ then $W_i \cap W_j^* \neq \emptyset$.*

PROOF. Let $x \in W_k \cap W_j^*$; apply 3.2 using the fact that $W_i \cap W_k^* \neq \emptyset$. Since W_k and W_j^* have normal intersection at x, it follows from 3.2 that $W_i \cap W_j^* \neq \emptyset$.

LEMMA 3.5. *Suppose $W_{i_k} \cap W_{i_{k+1}}^* \neq \emptyset$, $k = 1, \cdots, m$. Then $W_{i_k} \neq W_{i_j}$ if $j \neq k$.*

PROOF. First note by 3.1, dim $W_{i_{k+1}} \leq$ dim W_{i_k} and equality occurs only if $\beta_{i_{k+1}}$ is a closed orbit. This implies we can restrict ourselves to the case of the lemma where all the W_{i_k}'s are of the same dimension. Then if $W_{i_k} = W_{i_j}$, $k \neq j$, 3.4 implies that $W_{i_k} \cap W_{i_j}^* \neq \emptyset$. This contradicts condition (5).

LEMMA 3.6. *If $\partial W_\gamma \cap W_\delta \neq \emptyset$, then there is a sequence $W_{i_1}, \cdots, W_{i_m}$ such that $W_{i_k} \cap W_{ik+1}^* \neq \emptyset$, $W_\gamma = W_{i_1}$, and $W_\delta = W_{i_m}$.*

PROOF. Let $\alpha(W_\delta^*) = \lim_{t\to-\infty} W_\delta^*$. Then it follows that Cl $W_\gamma \cap \alpha(W_\delta^*) \neq \emptyset$. Let $\beta_j \in$ Cl $W_\gamma \cap \alpha(W_\delta^*)$. Then $W_j \cap W_\delta^* \neq \emptyset$. If $j \neq \gamma$, similarly let $\beta_k \in$ Cl $W_\gamma \cap \alpha(W_j^*)$. Induction and 3.5 yield 3.6.

LEMMA 3.7. *If $\partial W_i \cap W_j \neq \emptyset$, then $\partial W_i \supset W_j$ and either* dim W^i $>$ dim W_j *or* dim W_i = dim W_j, *$W_i \cap W_j^* \neq \emptyset$, and β_j is a closed orbit*

This follows from 3.6, 3.5, 3.3, 3.1, and 3.4.

LEMMA 3.8. *Each W_i is an imbedded R^p or $R^{p-1} \times S^1$.*

This follows from §2, 3.7 and (5).

LEMMA 3.9. *$\partial^k W_i \neq \emptyset$, any i for large enough k.*

If not there is a W_j such that $\partial^m W_j \cap W_j \neq \emptyset$. By 3.7 then $\partial W_j \cap W_j \neq \emptyset$, contradicting 3.8.

4. **On Morse theory.** A version of one of the standard theorems of Morse theory is stated in this section. The proof is a short well-known argument using the exact cohomology sequence of a pair and for example can be found in [**10**].

THEOREM 4.1. *Let M be an n-dimensional topological space with closed subspace L^p for each integer p such that $L^p \supset L^{p-1}$, and there* *exist integers a, b with* $L^a=\varnothing$ *and* $L^b=M$. *Using any fixed cohomology theory and coefficient field, assume dimension* $H^q(L^p, L^{p-1})$ *is finite for each p and q. Let* $B_q=\dim H^q(M)$ *and* $M_q=\sum_{r=a}^{b}\dim H^q(L^r, L^{r-1})$. *Then* M_q *and* B_q *satisfy the Morse relations*

$$M_0 \geqq B_0,$$

$$M_1 - M_0 \geqq B_1 - B_0,$$

$$M_2 - M_1 + M_0 \geqq B_2 - B_1 + B_0,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\sum_{k=0}^{n}(-1)^k M_k = \sum_{k=0}^{n}(-1)^k B_k.$$

5. **Proof of the main theorem.** Define $K^p=\bigcup_{\dim W_i\leq p} W_i$. Thom in [14] considers subspaces related to K^p to prove the classical Morse inequalities. By 3.7 it follows that the K^p are closed sets. However, examples show that K^p has the following bad property. It may be that for $W_i\subset K^p$, ∂W_i is not contained in K^{p-1}. To avoid this we define a new structure on M.

We define by induction a sequence of closed subsets L_i of M with $L_i\supset L_{i-1}$ and $L_i=M$ for large enough i. Define $L_0=\varnothing$, and if L_{i-1} has been defined, let L_i be the union of all the W_j whose boundary lies in L_{i-1}. It is immediate that $L_i\supset L_{i-1}$, that L_i is closed in M and that L_i-L_{i-1} consists of a disjoint union of W_j. It follows from 3.9 that there is an integer b such that $L_b=M$.

One can construct an example to show that the L_i need not be locally connected and that for $W_0\subset L_i-L_{i-1}$, $W_1\subset L_{i-1}$, $\dim W_0=\dim W_1$.

LEMMA 5.1. *Using Čech theory if* M_q *is as in* 1.1 *then*

$$M_q = \sum_{i=0}^{b}\dim H^q(L_i, L_{i-1}).$$

PROOF. As noted previously L_i-L_{i-1} consists of a disjoint union of the W_j and as i ranges from 0 to b all the W_j are obtained. Denoting cohomology with compact carriers by K^q_K, since $H^q_K(P-Q)=H^q(P,Q)$ for Čech theory, we have

$$\sum_{i=0}^{b}\dim H^q(L_i, L_{i-1}) = \sum_{\text{all } W_j}\dim H^q_K(W_j).$$

Using 3.8 and Poincaré duality $H^q_K(W_j)=H_{\dim W_j-q}(W_j)$. Furthermore $\dim H_0(W_j)=1$ for all j, $\dim H_1(W_j)=1$ if β_j is a closed orbit and $\dim H_p(W_j)=0$ otherwise. The lemma follows.

Theorem 1.1 follows from 4.1 and 5.1.

6. An analogue of the main theorem. Suppose instead of a vector field X on M, we just have given a C^∞ diffeomorphism h on M which satisfies certain conditions analogous to (1)–(5).

(1′) There are a finite number of periodic points (i.e., $x \in M$ such that $h^p(x) = x$ for some integer p) of h of simple type (i.e., the differential of h at p has no eigenvalue of absolute value 1).

(3′) The limit points of all the orbits of h (i.e., $\{h^p(x) \mid \text{all integers } p\} = \text{orbit of } x$) are periodic points.

(4′) The "stable" and "unstable" manifolds of the periodic points have normal intersection.

The previous theory extends to cover this case. In particular if M_q is the number of periodic points with q eigenvalues having absolute value greater than one, the Morse relations of 1.1 hold.

One can ask the corresponding questions of (A) and (B) of §1 for the above situation.

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