THE EQUIVALENCE OF FIBER SPACES AND BUNDLES[1]

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1. **Introduction.** The objective of this paper is to verify the conjecture made in [2] that every Hurewicz fibration [3] over a polyhedral base is fiber homotopy equivalent to a Steenrod fiber bundle [6]. The result relies heavily on Milnor's universal bundle construction [4] and the following extension [2] of a theorem of A. Dold [1].

THEOREM. *If $\{E_1, p_1, X\}$ and $\{E_2, p_2, X\}$ are Hurewicz fibrations over a connected CW-complex X and if $f: E_1 \to E_2$ is a fiber-preserving map such that f restricted to some fiber is a homotopy equivalence, then f is a fiber homotopy equivalence.*

2. **The associated bundle.** Let $\pi: E \to X$ denote a map, where X is a connected, locally finite polyhedron. Furthermore following Milnor's notation in [4], let $\tilde{S}$, $\tilde{E}$, $\tilde{G}$ denote, respectively, the simplicial paths in X, the simplicial paths emanating from a fixed vertex v_0 and the simplicial loops at v_0. If $\alpha = [x_n, \cdots, x_0]$ is a simplicial path in X we will find it convenient to set $\alpha(0) = x_0$, $\alpha(1) = x_n$. Now, define

$$\Omega_\pi = \{(e, \alpha) \in E \times \tilde{S} \mid \pi(e) = \alpha(0)\}$$

and a map $\xi: \Omega_\pi \to X$ by

$$\xi(e, \alpha) = \alpha(1).$$

Furthermore, let

$$A = \xi^{-1}(v_0) = \{(e, \alpha) \mid \pi(e) = \alpha(0), \alpha(1) = v_0\}.$$

LEMMA. *$\{\Omega_\pi, \xi, X, A, \tilde{G}\}$ is a Steenrod fiber bundle.*

PROOF. Since the proof is entirely analogous to Milnor's proof [4] that $\tilde{E}$ is a bundle over X, we content ourselves with a brief outline. The action $\mu: \tilde{G} \times A \to A$ is defined as follows:

$$\mu[g, (e, \alpha)] = (e, g\alpha).$$

Now, let v_j denote a vertex in X and V_j the star neighborhood of v_j. The coordinate functions

$$\phi_j: V_j \times A \to \xi^{-1}(V_j)$$

are defined by

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$$\phi_j(x, (e, \alpha)) = (e, [x, v_j]e_j\alpha)$$

where e_j is a fixed simplicial path from v_0 to v_j. We leave the remaining details to the reader.

Now, define $f: E \to \Omega_\pi$ by

$$f(e) = (e, [\pi(e), \pi(e)]).$$

The following diagram is easily seen commutative:

$$\begin{array}{ccc} E & \xrightarrow{f} & \Omega_\pi \\ & \pi\searrow \quad \swarrow\xi & \\ & X & \end{array}$$

3. **The equivalence theorem.** Let $\pi: E \to X$, $f: E \to \Omega_\pi$ be as in §2.

THEOREM. *If $\{E, \pi, X\}$ is a Hurewicz fibration, then f is a fiber homotopy equivalence.*

PROOF. Let $F = \pi^{-1}(v_0)$ denote the fiber in E over v_0. Then, in view of the theorem mentioned in the introduction, it suffices to show that $f' = f | F: F \to A$ is a homotopy equivalence.

Let $\overline{X}$ denote the space of ordinary paths in X ending at v_0 and let $\eta: \overline{X} \to X$ denote the fiber map given by $\eta(\alpha) = \alpha(0)$. Furthermore, let $\overline{E}$ denote the space of simplicial paths $[x_n, \cdots, x_0]$ on X such that $x_n = v_0$. Then, since $\overline{E}$ is homeomorphic to $\tilde{E}$ under the correspondence $[x_n, \cdots, x_0] \leftrightarrow [x_0, \cdots, x_n]$, $\overline{E}$ is a fiber bundle over X with fiber map $p: \overline{E} \to X$, given by $p([x_n, \cdots, x_0]) = x_0$ and fiber $\tilde{G}$. Consider then the fiber-preserving map $\bar{h}$

$$\begin{array}{ccc} \overline{X} & \xrightarrow{\bar{h}} & \overline{E} \\ & \eta\searrow \quad \swarrow p & \\ & X & \end{array}$$

defined as follows: Let $\bar{\lambda}$ denote a regular lifting function for $\{\overline{E}, p, X\}$ and if $\alpha \in \overline{X}$, set $\bar{\alpha}(t) = \alpha(1-t)$, $0 \leqq t \leqq 1$. Finally, define

$$\bar{h}(\alpha) = \bar{\lambda}([v_0, v_0], \bar{\alpha})(1).$$

Now, $\overline{X}$ and $\overline{E}$ are contractible, $\eta^{-1}(v_0) = \Omega(X)$, the space of ordinary loops on X, is dominated by a CW-complex and $\tilde{G}$ is a CW-complex. Therefore $\bar{h}$ restricted to $\Omega(X)$ is a homotopy equivalence and we may conclude that $\bar{h}$ is a fiber homotopy equivalence. Thus $\bar{h}$ possesses a fiber homotopy inverse h. If $\bar{v}_0 \in \overline{X}$ is the constant path and $[v_0, v_0] \in \tilde{G}$ is the identity in $\tilde{G}$, then $\bar{h}(\bar{v}_0) = [v_0, v_0]$ and h may be chosen so that $h([v_0, v_0]) = \bar{v}_0$. We employ h and $\bar{h}$ to define an auxiliary map $\chi: A \to A$ as follows. Define

$$\chi(e, \alpha) = (e, \bar{h}h(\alpha)).$$

Since $\bar{h}h$ is fiber homotopic to the identity map $\overline{E} \to \overline{E}$, $\chi \sim 1: A \to A$.

Next, we define a homotopy $H: A \times I \to A$. If ω is an ordinary path in X and $0 \leqq s, t \leqq 1$, set

$$\omega_s(t) = \omega(st)$$

and

$$\omega^s(t) = \omega(s + t - st).$$

Then, define, for $0 \leqq s \leqq 1$,

$$H((e, \alpha), s) = \{\lambda(e, [h(\alpha)]_s)(1), \bar{h}([h(\alpha)]^s)\}$$

where λ is a regular lifting function for $\{E, \pi, B\}$. Note that $\pi\lambda(e, [h(\alpha)]_s)(1) = h(\alpha)(s) = \bar{h}([h(\alpha)]^s)(0)$ since $\bar{h}$ preserves end points and $[h(\alpha)]^s(0) = h(\alpha)(s)$. Also $\bar{h}([h(\alpha)]^s)(1) = v_0$ for the same reason. Thus, $H((e, \alpha), s) \in A$. Furthermore,

$$H_0(e, \alpha) = (e, \bar{h}h(\alpha)) = \chi(e, \alpha),$$
$$H_1(e, \alpha) = \{\lambda(e, h(\alpha))(1), [v_0, v_0]\}$$

where $[v_0, v_0]$ is the identity in $\tilde{G}$.

Finally, we define the required homotopy inverse for $f': F \to A$. Set

$$g(e, \alpha) = \lambda(e, h(\alpha))(1).$$

Then, if $y \in F$,

$$gf'(y) = g(y, [v_0, v_0]) = \lambda(y, \bar{v}_0)(1) = y$$

and hence $gf' = 1$. Also, if $(e, \alpha) \in A$,

$$f'g(e, \alpha) = (\lambda(e, h(\alpha))(1), [v_0, v_0]) = H_1(e, \alpha).$$

Therefore $f'g \sim \chi \sim 1$ and g is a homotopy inverse for f'. This proves the equivalence theorem.

REMARK. It is not difficult to check that F considered as a subset of A is actually a strong deformation retract of A.

REMARK. It is quite clear that our main result is false for Serre fibrations [5] since there exist Serre fibrations over the unit interval whose fibers are not of the same homotopy type. Also, it is possible to exhibit examples of Hurewicz fibrations with 0-connected but not locally contractible base spaces for which our main result is false.

4. Extensions. The Equivalence Theorem is also valid if the base space X is *dominated* by a locally finite polyhedron. Thus, our main result can be stated as follows.

THEOREM. *Every Hurewicz fibration over a base space dominated by a locally finite polyhedron is fiber homotopy equivalent to a Steenrod fiber bundle.*

An interesting application is the following corollary.

COROLLARY. *If X is a connected space dominated by a locally finite polyhedron, then for every integer $n \geqq 1$, there exist n-connective Steenrod fiber bundles over X.*

PROOF. One merely applies the above theorem to the n-connective Hurewicz fibrations over X given by G. W. Whitehead in [7].

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