

NEW RESULTS AND OLD PROBLEMS IN FINITE TRANSFORMATION GROUPS¹

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We shall be concerned with the topology of finite transformation groups on spaces of relatively simple character. This represents a rather small corner in the general theory, but the problems which one finds here seem to be of some interest and difficulty. By disposing of various low-dimensional cases, we shall try to show where the real difficulties begin.

1. Definitions. A transformation group (G, X) consists of a group G acting on a topological space X to form a group of homeomorphisms of X onto itself. It will be understood throughout this paper that G is finite. For a given transformation group or "action" (G, X) and subset $H \subset G$, we denote by $F(H; G, X)$ the fixed-point set of H —that is, the points x such that $hx = x$ for $h \in H$. We may of course denote this set simply by $F(H)$ when only one action is being considered. An action (G, X) is *g-free* if $F(g) = \emptyset$, *free* if it is *g-free* when $g \neq 1$, and *semi-free* if $F(G) = \emptyset$. Let X^r be the union of the fixed-point sets $F(g)$, $g \neq 1$, and let $X^f = X - X^r$. Since $gF(h) = F(ghg^{-1})$, the closed set X^r is invariant under the transformations $x \rightarrow gx$. Hence G acts on X^r (if X^r is not empty) hence also on X^f . The action (G, X^f) is free but the action (G, X^r) is not free. We call X^f the *free part* of X and X^r the *restricted part*; (G, X^r) may be called the restricted part of the action (G, X) . An action (G, X) is *effective* if $F(g) \neq X$ when $g \neq 1$. In a given action, the set N of elements g with $F(g) = X$ is a normal subgroup of G and there is induced an action $(G/N, X)$ which is effective.

The sets Gx , $x \in X$, are the *orbits* of (G, X) . They form a decomposition of X and the corresponding decomposition space, called the *orbit space* of the action, is denoted by X/G . The *stability group* G_x of x consists of all g such that $gx = x$.

An action (G, X) is of class C^k if X is a manifold of class C^k and the functions $x \rightarrow gx$ are of class C^k . When $k=0$ we shall drop the manifold condition on X ; every action is then of class C^0 . A *differentiable* action is a C^1 -action. (G, X) is *orthogonal* if X is a euclidean sphere or an open submanifold of a euclidean space and the trans-

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formations $x \rightarrow gx$ are orthogonal. An action (G, X) which is of class C^k (or orthogonal) is *triangulated* if X carries a triangulation which is of class C^k (or orthogonal) and is compatible with the action. An orthogonal triangulation here means one in which the cells are simplexes if X is a euclidean space, and geodesic simplexes if X is a euclidean sphere.

2. Isomorphism classes. An *isomorphism* $\phi: (G, X) \rightarrow (G', X')$ of two actions, both of class C^k or orthogonal, consists of an isomorphism $G \rightarrow G'$ and a homeomorphism $X \rightarrow X'$, of class C^k or orthogonal (both mappings being bijective) such that $gx \rightarrow g'x'$ whenever $g \rightarrow g'$, $x \rightarrow x'$. For a given G and X , the actions (G, X) of class C^k fall into isomorphism classes the totality of which we denote by $I^k(G, X)$. Let $I^{or}(G, X)$ be the corresponding sets for orthogonal actions. Since every action and isomorphism of class C^k can be regarded as of class C^0 , we have a natural mapping $I^k(G, X) \rightarrow I^0(G, X)$. Similarly we have $I^{or}(G, X) \rightarrow I^0(G, X)$.

For each G , the mappings

$$I^{or}(G, S_n) \rightarrow I^k(G, S_n) \rightarrow I^0(G, S_n)$$

are bijective when $n = 0, 1, 2$. This is trivial for $n = 0$ and easy for $n = 1$; the proof for $n = 2$ is due to Kérékjártó [12].

The mappings $I^{or}(Z_2, S_n) \rightarrow I^0(Z_2, S_n)$ are not all surjective when $n \geq 3$. Bing [1], for example, has constructed an action (Z_2, S_n) in which $F(Z_2)$ is a topological 2-sphere, and the free part of S_3 is not the disjoint union of two 3-cells as it would have to be were the action isomorphic to an orthogonal one. Actions (Z_2, S_n) not isomorphic to orthogonal ones and with arbitrarily large n , have been given by J. H. C. Whitehead [25]. On the other hand, Hirsch and Smale [10] have shown that every action (Z_2, S_3) which admits exactly two fixed points² is isomorphic to an orthogonal one and G. R. Livesay (not yet published) showed that the same is true of every free (Z_2, S_3) . An example of Floyd [7] showed that $I^{or}(Z_6, S_{41}) \rightarrow I^0(Z_6, S_{41})$ is not surjective, in fact the image of I^{or} does not even contain all members of I^0 which have triangulable representatives. In the triangulated action (Z_6, S_{41}) constructed by Floyd, $F(Z_6)$ is not homeomorphic to a euclidean sphere.

Call two orthogonal actions (G, S_n) *combinatorially isomorphic* if they admit triangulations which are carried one into the other by

² Other supports can be used. Concerning the definitions and statements in this paragraph see [18] or [1; 19; 20].

some (not necessarily orthogonal) isomorphism. It is easy to see that orthogonal isomorphism implies combinatorial isomorphism and hence we have a natural mapping $I^{\text{or}}(G, S_n) \rightarrow I_c^{\text{or}}(G, S_n)$ where I_c^{or} consists of the combinatorial isomorphism classes. These mappings are injective when $G = Z_m$, $m > 1$. This was proved by de Rham [16] for free actions and the general case follows readily.

Not much more is known at present about the natural mappings of isomorphism classes.

3. Joins. Two transformation groups (G, X) , (G', X') determine in a natural way an action $(G \times G', X \circ X')$ on the *join* $X \circ X'$. If $G = G'$ we may restrict the action of $G \times G$ to the diagonal of $G \times G$ and obtain an action $(G, X \circ X')$. We shall denote this action by $(G, X) \circ (G, X')$.

The join $S_m \circ S_n$ of two euclidean spheres is homeomorphic to S_{m+n+1} . In fact one can assign the structure of a euclidean $(m+n+1)$ -sphere to $S_m \circ S_n$ in such a way that $(G, S_n) \circ (G, S_m)$ is orthogonal and uniquely determined up to (orthogonal) isomorphism. The multiplication defined by $(G, S_n) \circ (G, S_m)$ is associative up to isomorphism.

Suppose that G is abelian and (G, S_n) orthogonal. Then from elementary properties of real representations there is a "decomposition"

$$(1) \quad (G, S_n) = (G, S) \circ (G, S') \circ \cdots \circ (G, S^{(k)})$$

where S, S', \dots are spheres and where the factors are orthogonal and can not themselves be factored; this decomposition is unique up to isomorphism and the order of the factors. The transformation groups and isomorphism in (1) can of course be regarded as being of class C^0 , but from the C^0 point of view it is not known whether uniqueness holds. The question of C^0 -uniqueness for the decomposition of *free* orthogonal actions (Z_m, S_{2n+1}) , $m > 2$, is essentially equivalent to the problem of classifying lense spaces, which are the orbit spaces of such actions. De Rham [16] showed that uniqueness does hold for such actions if "isomorphism" is taken to mean "combinatorial isomorphism." This implies a combinatorial classification of lense spaces.

The examples of Bing, Floyd and Whitehead in §2 show that not every abelian (G, S_n) (of class C^0) has a decomposition (1) even if it is triangulable. It is in fact doubtful that a decomposition necessarily exists in the differentiable case. For one reason, the sets $F(g)$ for the right member of (1) with differentiable factors, are homeomorphic

to spheres whereas it is doubtful if this is true of the set $F(g)$ for every differentiable action (G, S_n) although so far as the writer knows, no counter-example has been constructed.

4. Effective actions in spheres. Let $E^k(X)$ denote the totality of finite groups G such that there exist effective actions (G, X) of class C^k , and let $E^{\text{or}}(X)$ denote the corresponding set for orthogonal actions. It is known that $E^0(S_n) = E^{\text{or}}(S_n)$ when $n = 0, 1, 2$ but it is not known whether this is true when $n > 2$. (We shall consider the case $n = 2$ in §7.) It should be remarked that while the members of $E^{\text{or}}(S_n)$ can be listed when $n \leq 3$ (Seifert and Threlfall [17]), this is not the case when $n > 3$. We note also that *every* group G can act effectively on some S which has the same homotopy type as S_n . In fact let $S = S_n \cup (p \circ G)$ where p is a point of S_n and let (G, S) be defined by $gx = x$, $x \in S_n$, $g(p \circ h) = p \circ gh$. This action is effective, and S is retractible to S_n by deformation.

No example is known of an effective action (G, S_n) where G can *not* act effectively and *orthogonally* on S_n , and one might conjecture that no such action exists. At any rate, one can show that *none exists in which G is an abelian p -group*. This is a straightforward consequence of the next proposition.

By a *homology n -sphere* over A we shall mean a locally compact finite dimensional Hausdorff space X such that $H^*(X, A) = H^*(S_n; A)$ where H^* means cohomology with compact supports.² It is to be understood that S_0 consists of two points and that the empty set is a homology (-1) -sphere. If X is a homology n -sphere over Z_p , p a prime, and if Z_p acts on X , then $F(Z_p)$ is a homology m -sphere over Z_p and $-1 \leq m \leq n$; $n - m$ is even if $p > 2$. The example above shows that m can equal n even if the action is effective, but this possibility can be ruled out by imposing local conditions. Call X a *generalized n -sphere* over a principal ideal domain A if it is a homology n -sphere over A and a generalized n -manifold over A . A generalized 0-sphere over A consists of two points. If X is a generalized n -sphere over Z_p , then for any action (Z_p, X) , $F(Z_p)$ is a generalized m -sphere over Z_p and if the action is effective, $m < n$. If X is a homology n -sphere over Z_2 and $H^n(X; Z)$ is finitely generated, one can distinguish between actions (Z_2, X) which preserve orientation and those which reverse orientation. If orientation is preserved then $n - m$ is even.

Let X be a generalized n -sphere over Z_p , p a prime, and let $G = Z_p \times \cdots \times Z_p$, s factors, act effectively on X . If $p > 2$, then $s \leq (n + 1)/2$. In any case $s \leq n + 1$.

Suppose $p > 2$. If $s = 1$, there is nothing to prove. Assume $s \geq 2$. We shall show that there exist subgroups $G = G^1 \supset G^2 \supset \cdots \supset G^{s-1}$, each

of index 1 in the preceding, and generalized spheres $X = X^n, X^{n-1}, \dots, X^{s-1} \neq \emptyset$ of strictly decreasing dimension such that G^i acts on X^i (i.e. $gX^i = X^i$ whenever $g \in G^i$) and does so effectively. Suppose in fact that for some i with $i < s-1$, G^1, \dots, G^i and X^1, \dots, X^i have been defined. The relation $i < s-1$ implies that G^i is noncyclic and therefore (G^i, X^i) can not be free [22]. Hence there exist cyclic subgroups H of G^i such that $F(H) (= F(H; G^i, X^i))$ is nonempty. Among the cyclic subgroups H of G^i for which $F(H) \neq \emptyset$, let H^1 be one such that $F(H^1)$ is maximal i.e. is not properly contained in some $F(H)$. Take G^{i+1} to be a nontrivial cyclic subgroup of G^i such that $G^{i+1} \cap H^1 = \{1\}$, and take X^{i+1} to be $F(H^1)$ noting that $X^{i+1} \neq \emptyset$. Since G is abelian, G^{i+1} acts on X^{i+1} and it only remains to be proved that (G^{i+1}, X^{i+1}) is effective. If not, there is a nontrivial cyclic subgroup H' of G^{i+1} leaving X^{i+1} pointwise fixed. This means that the fixed-point set of H' , acting on X^i , contains X^{i+1} , i.e. that $F(H') \supset F(H^1)$. Hence by maximality, $F(H') = F(H^1)$ and therefore the nontrivial elements of $H'H^1$ all have the same fixed-point set, namely $F(H^1)$ and therefore [22] $H'H^1$ is cyclic, hence $H' = H^1$. But this is impossible since $H' \subset G^{i+1}$. Now the X 's are generalized spheres over Z_p , and each is the fixed-point set of an action on the preceding by a cyclic subgroup of G . The "dimensions" of the X 's form a strictly decreasing sequence ending with that of X^{s-1} , call it k . Then $0 \leq k \leq n - (s-1)$, hence $x \leq n+1$. If $p > 2$, the dimensions decrease by *even* jumps, hence $0 \leq k \leq n - 2(s-1)$, $s \leq (n+1)/2$.

5. Groups of order pq . Let $G(p, q, k)$ denote the nonabelian group of order pq defined by $s^q = t^p = 1$, $sts^{-1} = t^k$, with p, q distinct primes, $p > 2$, $k \neq 1$, $k^q = 1 \pmod p$.

Call an action (G, X) *regular* if the sets $F(g)$, $g \in G$, are manifolds. If (G, X) is regular and X is orientable, all the manifolds $F(g)$ are orientable [20]. Consider an action (G_{pq}, S_n) where $G_p = G(p, q, k)$. Let S_n and $F(t)$ be oriented. G_{pq} acts on $F(t)$ since the subgroup generated by t is normal. Call the action (G_{pq}, S_n) *concordant* if s preserves or reverses both orientations, agreeing that the orientation of F is preserved if $F = \emptyset$. The action is necessarily concordant if $q > 2$. Let $r = \dim F(t)$ agreeing that $\dim \emptyset = -1$. Since $p > 2$, $n-r$ is even. A regular s -free action in which $q = 2$ is necessarily concordant.

Suppose $G(p, q, k)$ acts regularly, effectively and concordantly on S_n . Then $n \equiv r \pmod{2q}$. The hypothesis of regularity can be omitted if the action is t -free.

PROOF. Let $F = F(t)$, and let S_n and F be oriented. To each trans-

formation $T: S_n \rightarrow S_n$ of period p having F as fixed-point set, there is associated [11], Appendix B an element of Z_p , denoted by $\text{ind } T$, such³ that (1) $\text{ind } T^k = k^{(r-n)/2} \text{ind } T$ where $r = \dim F$; (2) if S is a homeomorphism $(S_n, F) \rightarrow (S_n, F)$ which preserves or reverses both orientations, then $\text{ind } (STS^{-1}) = \text{ind } T$. Let $\text{ind } t = \text{ind } T$ where T is the transformation $x \rightarrow tx$. Then

$$\text{ind } t = \text{ind } sts^{-1} = \text{ind } t^k = k^{(r-n)/2}$$

so that $k^{(r-n)/2} = 1 \pmod p$. Since $k^q = 1 \pmod p$ and $k \neq 1$, we see that q must divide $(r-n)/2$, hence $r-n = 0 \pmod{2q}$.

The preceding proposition restricts the cases in which effective action can occur. We mention the following instances:

- (a) If $q > 2$, $G(p, q, k)$ can act t -freely on S_n only if $n+1 = 0 \pmod{2q}$.
- (b) If $q > 2$, $G(p, q, k)$ can not act effectively and regularly on S_2, S_3, S_4 . If there exists such an action on S_6 it must be t -free.
- (c) If $q > 2$, $G(p, q, k)$ can not act differentiably and effectively on S_6 . For, suppose there is such an action. Since $6-r \geq 2q \geq 6$, and $6-r$ is even, we have $r=0$. Hence $F(t)$ consists of two points. Since q is odd, both points are fixed under s . Hence $F(G_{pq}) \neq \emptyset$. (G_{pq}, S_6) induces an effective orthogonal action on the tangent vector space of any point of $F(G_{pq})$ hence an effective regular action of (G_{pq}, S_6) , which is impossible by (b).

If $q=2$, then $k=-1$ and $G(p, 2, -1)$ is the dihedral group G_{2p} of order $2p$. Milnor [13] showed that G_{2p} *can not act freely on any S_n* . Of course G_{2p} can not act freely and *orthogonally* on S_n for the reason that since s is of order 2 and $F(s)$ is empty, s would be represented by the matrix $-I$ and would therefore permute with t .

In the cases which have not been excluded by the preceding remarks, it is not known whether $G(p, q, k)$ can act effectively on a given sphere. As Milnor remarked, the simplest unsolved case for free actions is the following. Can $G(7, 3, 2)$ of order 21 act freely on S_6 ? Zassenhaus [28] showed that $G(p, q, k)$ can not act freely and orthogonally on any S_n .

In the case of actions which are not free, perhaps the simplest unsolved case is the following. Suppose (G_{2p}, S_4) is regular and concordant. Then $4=r \pmod 4$, hence $r=0$. Can this actually occur? Specifically, can the dihedral group of order 6 (i.e. $G(3, 2, -1)$) act on S_4 in such a way that $F(s) = \emptyset$ and $F(t)$ consists of two points? No such orthogonal action is possible.⁴

³ (1) is not explicitly stated in [11] but can easily be verified.

⁴ A related question: let (G, S_n) be a differentiable action such that $F(G)$ consists of two points x, y . Are the induced actions $(G, T_x), (G, T_y)$ on the tangent vector spaces isomorphic?

Most of the results in this section remain true if S_n is an n -manifold which is an integral homology n -sphere. It remains true for example that the dihedral group can not act freely on S_n . On the other hand, recent results of Swan [24] show that such actions do exist on homology spheres which are not manifolds.

6. Groups with periodic cohomology. It is well known [3, p. 358] that if G acts freely on a sphere of odd dimension n , then G has cohomology of period $n+1$. This means that [3, p. 260] $\hat{H}^{n+1+k}(G; A) = \hat{H}^k(G, A)$, $k=0, 1, \dots$ for every coefficient module A . The only properties of the \hat{H}^k which we shall need are the following [3, p. 237, p. 250]:

$$\begin{aligned}\hat{H}^0(G, Z) &= Z_d, & d &= [G: 1], \\ \hat{H}^2(G, Z) &= \text{Hom}(G/[G, G], Z_d).\end{aligned}$$

It is assumed in these formulas that G acts trivially on Z .

REMARK. If $\hat{H}^0(G, Z) = \hat{H}^2(G, Z)$ with trivial action on Z , then $G = Z_d$. For, $\text{Hom}(G/[G, G], Z_d) = Z_d$ implies easily that $G/[G, G]$ contains an element of order d and this in turn implies that $G = Z_d$ since G and Z_d are both of order d .

We shall need the following mild generalization of the theorem of periodic cohomology.

Let X be an integral homology n -sphere and $Y \subset X$ an integral homology m -sphere with $m \leq n-2$. If G acts on X in such a way that $gY = Y$ for $g \in G$ and G acts freely on $X' = X - Y$, then G has cohomology of period $n-m$.

PROOF. There exists a spectral sequence E_r [3, p. 354] or [18, Chapter IV] such that E_∞ is associated with the cohomology of the orbit space X'/G over a coefficient module A and $E_2^{p,q} = H^p(G, H^q(X', A))$. Cohomology here is taken with compact supports and is reduced in the dimension 0. From the Künneth relations, $H^n(X, A) = A$, $H^h(X, A) = 0$, $h \neq n$. Hence from the cohomology sequence for (X, Y) we have $H^*(X', A) = H^n(X', A) \oplus H^{m+1}(X', A) = A \oplus A$. Thus $E_2^{p,q}$ is nontrivial only when $q = n, m+1$. Using $E_{r+1} = H(E_r)$ and the fact that d_r is of bi-degree $(r, 1-r)$, we find that $E_2^{sn} = \dots = E_r^{sn}$ and $E_2^{s, m+1} = \dots = E_r^{s, m+1}$ where $r = n-m$, $s = 0, 1, \dots$. We have

$$(1) \quad d_r^{s,n} : E_r^{s,n} \rightarrow E_r^{s+n-m, m+1} \quad (s = 0, 1, \dots).$$

We assert that $d_r^{s,n}$ is bijective when $s \geq 1$. For, $E_2^{p, m+1}$, $p \geq n-m$, evidently consists of permanent cocycles of total degree $\geq n+1$. But E_∞ gives the cohomology of X'/G which can be shown to be trivial

in dimensions exceeding n (in order not to stop for this point we can assume that $\dim X = n$ in which case it is trivial). Let $x \in E_r^{s+n-m, m+1}$. By what has just been said, the image of x in E_∞ is zero. Assume $x \neq 0$. Then x must be "killed off" in passage to ∞ which means that x has an image in some E_t , $t \geq r$, of the form $d_t y$, $y \neq 0$. Using the fact that d_t is of bidegree $(t, 1-t)$ we see that this can happen only if $t = r$ and $y \in E_r^{s, n}$. It follows that (1) is surjective. Suppose $s > 1$. If (1) is not injective $E_r^{s, n}$ would contain a nonzero permanent co-cycle, which, being of total degree $\geq n+1$, must be killed off by an element in some $E_s^{u, v}$ where $v > n$ whereas, all such elements being images of elements in $E_2^{u, v}$ are zero, which proves the assertion. It follows from this and the fact that $H^s = \hat{H}^s$ when $s \geq 1$, that $\hat{H}^s(G, A) = \hat{H}^{n-m+s}(G, A)$ for every A when $s \geq 1$. It must therefore hold also for $s = 0$ [3, p. 358].

Using the remark in the second paragraph of this section we have the following

COROLLARY. *If X is an integral homology n -sphere and (G, X) an action such that $X^f = X - F(G)$ and if $F(G)$ is an integral homology $(n-2)$ -sphere, then G is cyclic.*

7. Actions on a generalized 2-sphere. Let Ω be the set of orbits of an action (G, X) where X is finite (discrete). Since $G_{gx} = gG_xg^{-1}$ (§2) we see that $[G_x: 1]$ is constant on each orbit ω . Let $\nu(\omega) = [G_x: 1]_{x \in \omega}$. Let $N = [G: 1]$ and let $\phi(g)$ be the number of points in $F(g)$. Then

$$(1) \quad \sum_{g \neq 1} \phi(g) = N \sum_{\omega \in \Omega} (1 - 1/\nu(\omega)).$$

This is shown by counting the number P of pairs (g, x) such that $g \neq 1$, $gx = x$. There are $\phi(g)$ x 's paired with each $g \neq 1$, hence the left members of (1) equals P . For x in a given orbit ω , there are $\nu(\omega) - 1$ g 's paired with x ; hence to ω there correspond $(\nu(\omega) - 1)n(\omega)$ pairs where $n(\omega)$ is the number of elements in ω . Now $n(\omega)$ equals the number of cosets of G_x , $x \in \omega$, hence $n(\omega)\nu(\omega) = N$. Hence the number of pairs corresponding to ω is $N(\nu(\omega) - 1)/\nu(\omega)$ and therefore P equals the right member of (1).

Let G act on a finite set X such that $F(g)$ consists of a single point when $g \neq 1$ and each G_x is nontrivial. Then X consists of just one point.

For, with $\phi(g) = 1$, $g \neq 1$, (1) becomes

$$(2) \quad 1 - 1/N = \sum (1 - 1/\nu(\omega)).$$

$G_x \neq \{1\}$ implies $\nu(\omega) \geq 2$ for every ω . Hence the right member of (2) is > 1 if there are two or more terms in the sum whereas the left member is smaller than 1. We conclude that there is just one orbit ω

and that $\nu(\omega) = N$. Hence $G_x = G$ for each x so that $F(G) = X$. Therefore $F(g) = X$ for every G , hence X consists of one point.

Consider now the subset $E^{\text{rot}}(S_2)$ of $E^{\text{or}}(S_2)$ consisting of those groups which can act effectively as rotation groups on S_2 .

If there exists a (G, X) with finite X such that every G_x is cyclic and nontrivial and every $F(g)$, $g \neq 1$, consists of two points, then $G \in E^{\text{rot}}(S_2)$.

In fact, a classical argument [27, p. 17] based on (2) shows that if (G, X) has the stated properties then (G, X) can be identified with the restricted part (§1) of an effective action of G in S_2 , in which the transformations $x \rightarrow gx$ are rotations.

Let X be an integral homology 2-sphere. If the restricted set X^r of an action (G, X) is finite and each set $F(g)$, $g \in G$, is nonempty, then G is a member of $E^{\text{rot}}(S_2)$.

It is sufficient to show that the restricted part (G, X^r) satisfies the hypothesis of the preceding proposition. Let g be an element of G of prime order p . Then $F(g)$ is a homology 0-sphere over Z_p . But as a subset of X^r , $F(g)$ is finite and therefore consists of exactly two points. It follows readily that every $F(g)$, $g \neq 1$, consists of two points. Since each point in X^r is fixed under some $g \neq 1$, G_x is nontrivial when $x \in X^r$. It remains to be shown that G_x , $x \in X^r$, is cyclic. Let x be a point in X^r , and consider the action (G_x, X) . Let X'_x be the restricted part of X in this action. Evidently $x \in X'_x$ and G_x acts on the set $X' = X'_x - \{x\}$ which is nonempty. Each element of G_x different from 1 leaves just two points of X fixed, one of which is x , hence leaves one point of X' fixed. Moreover, if $x' \in X'$ then at least one g in G_x different from 1 leaves x' fixed. Therefore by the first proposition in this section X' consists of a single point and so in the action (G_x, X) , $F(G_x)$ consists of two points and is therefore an integral homology 0-sphere. Moreover, the free part of X in this action is $X - F(G_x)$. Hence by the corollary in §6, G_x is cyclic.

COROLLARY. *If G acts effectively on an integral generalized 2-sphere so that the transformations $x \rightarrow gx$ preserve orientation, then G is a member of $E^{\text{rot}}(S_2)$.*

8. Cyclic actions on 3-spheres. Let L be the restricted part of S_3 for an action (Z_m, S_3) and assume that L is the disjoint union of k simple closed curves, $k \geq 1$. If the action is orthogonal, k is either 1 or 2. This remains true if orthogonality is replaced by the hypothesis that each component J of L is unknotted in the sense that $\pi_1(S_3 - J) = Z$. Whether or not a component of L can actually be knotted is an open question; Montgomery and Samelson [14] have shown that certain types of knots are not possible.

Let $m = pqr$ where p, q, r are distinct odd primes and let $J_p = F(Z_p)$, etc. If J_p, J_q, J_r are disjoint simple closed curves, each is knotted.

PROOF. Z_m acts on each J . Suppose that J_p is unknotted: $\pi_1(U) = Z$, $U = S - J_p$, $S = S_3$. The universal covering \tilde{U} of U is a 3-manifold which is aspherical since U is aspherical [15], hence is acyclic with respect to integral homology. Hence the fixed-point set of any transformation $\tilde{U} \rightarrow \tilde{U}$ of odd prime period is a nonempty manifold of dimension r where $0 \leq r < 3$ and $3 - r$ is even; hence $r = 1$. The fixed-point set will moreover be connected and noncompact and is therefore a line, i.e. a topological image of E_1 . Let ϕ be the projection $\tilde{U} \rightarrow U$ and let $\tilde{J}_q = \phi^{-1}J_q$, $\tilde{J}_r = \phi^{-1}J_r$. Let $x \in J_q$. We can think of \tilde{U} as consisting of the equivalence classes of paths in U based at x . There is an obvious action of Z_q on \tilde{U} defined by the action of Z_q on these paths. Since J_q, J_r are invariant under Z_p , so are \tilde{J}_q, \tilde{J}_r (\tilde{J}_r for example consists of the equivalence classes of paths based at x and ending at points of J_r , and is therefore invariant). Now each component \tilde{K} of \tilde{J}_q is a line or a simple closed curve. But \tilde{K} can not be a s.c.c. because $\phi\tilde{K}$, which equals J_q , would be null-homotopic in U , hence in the action of Z_q on U , J_q would be an invariant s.c.c. which bounds in U , and no such curve exists.⁵ Now Z_p acts on the equivalence classes of loops in U based at x , namely on $\pi_1(U) = Z$ and since q is odd, this action is trivial. These loops can be thought of as giving the points \tilde{x} which cover x . Hence Z_q leaves each cover \tilde{x} of x fixed. Each component \tilde{K} of \tilde{J}_q contains an \tilde{x} and is therefore invariant under Z_q . Thus Z_q acts on \tilde{K} and since $q \neq 2$ and \tilde{K} is a line, the action is trivial. We conclude that each point of \tilde{J}_q is fixed under Z_q . Conversely, a point of \tilde{U} fixed under Z_q must cover a point of J_q , hence is in \tilde{J}_q . Thus \tilde{J}_q is the fixed-point set of Z_q acting on \tilde{U} , hence is a line. In the same way, \tilde{J}_r is a line. But Z_q acts on \tilde{J}_r and q being odd, the action is trivial. Hence \tilde{J}_r consists of fixed points of Z_q hence $\tilde{J}_r \subset \tilde{J}_q$ which implies $J_r \subset J_q$, a contradiction.

9. Acyclic spaces. No nontrivial finite group can act freely on a euclidean space or on a closed euclidean ball. The situation with regard to semi-free actions on such spaces stands about as follows. (a) Greever [9] showed that no group of order less than 60 and different from 36 and no abelian group of order less than 210 can act semi-freely on a closed ball; (b) Floyd and Richardson [8] showed that there exists a triangulable semi-free action of A_5 on a closed ball of

⁵ The existence of such a curve together with the fact that $H_n(U, Z_p) = 0$ for $n \geq 2$ would imply the existence of a fixed point for the action (Z_p, U) by essentially the same argument as used in the proof of Theorem I(α) in [21].

suitably high dimension, where A_5 is the group of even permutations on 5 letters, hence of order 60; (c) for $m \geq 2$, Z_m can not act semi-freely on E_n if $n \leq 4$; (d) there exists no differentiable semi-free action Z_{pq} on E_n if p, q are primes and $n \leq 6$; (e) there exists an orthogonal semi-free action of Z_{pq} on a contractible submanifold of E_n where n is a suitable multiple of pq (Conner and Floyd [4]).

We shall sketch the proofs of (c), (d) in this section and (e) in the next.

(c) By a one point compactification, a given action (Z_m, E_n) induces an action (Z_m, S_n) in which $\infty \in F(Z_m)$. It is sufficient to show that in any action (Z_m, S_n) with $n \leq 4$, $F(Z_m)$ can not consist of just one point. This is trivial for $n=0$, easy for $n=1, 2$ and straightforward for $n=3$. Consider an action (Z_m, S_4) . We may assume [21] that m is not the power of a prime. Then Z_m contains a subgroup Z_p where p is a prime different from 2. Then $F(Z_p)$ consists of two points or is a homology 2-sphere over Z_p . In the second case, enough local properties of $F(Z_p)$ can be established to ensure by a theorem of Wilder [26, Theorem 4.23, p. 223] that it is homeomorphic to S_2 . Thus $F(Z_p)$ is homeomorphic to S where $S=S_0$ or S_2 . Now Z_m acts on S and the fixed-point set of Z_m in S_4 is identical with the fixed-point set of Z_m in S_4 , hence can not consist of just one point.

(d) We shall sketch the proof of a proposition from which (d) follows readily:

Let Z_{pq} act differentiably on E_n and assume that $F(Z_p)$, which is a differentiable submanifold of E_n , is of dimension $n-2$. Then $F(Z_{pq}) \neq \emptyset$.

Let $F=F(Z_p)$. The closed differentiable $(n-2)$ -manifold F is orientable, connected, and noncompact. Duality relations show that $H_1(E_n - F; Z) = Z$. We consider the linking numbers $l(\xi, \Delta)$ where ξ is an arbitrary integral singular 1-cycle in $E_n - F$ and Δ an integral infinite fundamental $(n-2)$ -cycle for F . The integer $l(\xi, \Delta)$ depends only on the homology class of ξ in $H_1 = H_1(E_n - F, Z)$. If $t \in H_1$ and $g \in Z_p$, then $gt = \epsilon t$ when $|\epsilon| = 1$; -1 can only occur if $p=2$. Hence $l(g\xi, \Delta) = \epsilon l(\xi, \Delta)$ so in any case $l(g\xi, \Delta) = l(\xi, \Delta) \pmod p$, and $(\sigma\xi, \Delta) = 0 \pmod p$ where $\sigma = 1 + g + \dots + g^{p-1}$. Call a Z_p -invariant 1-cycle in $E_n - F$ *simple* if it is expressible as $\sigma\alpha$ where $\partial\alpha = x - gx$, x a point, $g \neq 1$. If η_1, η_2 are simple 1-cycles then $l(\eta_1, \Delta) = l(\eta_2, \Delta) \pmod p$. For say $\eta_i = \sigma\alpha_i$, $\partial\alpha_i = x_i - gx_i$. Let β be a 1-chain in $E_n - F$ such that $\partial\beta = x_2 - x_1$. Then $\eta_1 - \eta_2 = \sigma\zeta$ where ζ is the cycle $\alpha_1 - \alpha_2 + \beta - g\beta$. Hence $l(\eta_1 - \eta_2, \Delta) = 0 \pmod p$. Now Z_p acts on F and F is connected. Hence there are simple cycles in F . Let μ be one. If $F \cap F(Z_q) = \emptyset$, then $\mu \subset E_n - F$. On the other hand, $H_1(F(Z_q), Z_q) = 0$ and therefore

the homology class of μ (or any integral 1-cycle in $E_n - F$) is divisible by arbitrarily large multiples of p . The same is true of $l(\mu, \Delta)$ which implies that $l(\mu, \Delta) = 0$. On the other hand, it follows easily from differentiability that there exist simple cycles β in $E_n - F$ near F such that $l(\beta, \Delta) = 1$. Then since $l(\mu - \beta, \Delta) = 0 \pmod p$ we have $l(\beta) = 0 \pmod p$ which is impossible. It follows that $F \cap F(Z_q)$, which equals $F(Z_{pq})$, is not empty.

QUESTION. Can $F(Z_{pq})$ be compact? ($F(Z_p)$ and $F(Z_q)$ are not.)

10. Semi-free actions. Let X_q be finite dimensional and acyclic with respect to homology over Z_q . Let p, q be primes. Although there exists no semi-free action (Z_p, X_q) when $p = q$, the following example shows that such actions may exist when $p \neq q$.

Let C be a circle with angular coordinate θ and $\{m^k\}$ the mapping $C \rightarrow C$ of degree m^k given by $\theta \rightarrow m^k \theta$. Let ρ_m be the rotation $\theta \rightarrow \theta + 2\pi/m$. For $t \geq 0$ let $[t]$ be the largest integer $\leq t$. On the semi-infinite cylinder $W^2 = C \times [0, \infty)$ the sets $(\{m^{[t]}\}^{-1}c, t)$, $c \in C$, form a decomposition defining an equivalence ϵ_m . Let $W_m^2 = W^2/\epsilon_m$ and consider the subsets $W(t) = (C \times t)/\epsilon_m$, $W[t, s] = (C \times [t, s])/\epsilon_m$ etc. of W_m^2 . For $k = 0, 1, \dots$, $W[k, k+1]$ is retractible to $W[k+1]$ by a deformation which maps every $W(t)$, $t \in [k, k+1)$ onto $W(k+1)$ with degree m . It follows readily that W_m^2 is acyclic over Z_m . The rotation $(c, t) \rightarrow (\rho_n c, t)$ of W^2 induces an action (Z_n, W_m^2) which is obviously free if m is prime to n , as we now suppose. An easy triangulation makes this action simplicial. W_m^2 , which is now an infinite 2-complex can be imbedded in $E_j \times \dots \times E_j$ (n factors, j sufficiently large) in such a way [7] that (Z_n, W_m^2) is induced by a simplicial orthogonal action $(Z_n, E_j \times \dots \times E_j)$ where $g(x, y, \dots, z) = (y, \dots, z, x)$, g a generator of Z_n . The group Z_n acts freely on the regular neighborhood U of W_n^2 and the action (Z_n, U) is orthogonal (§1). Since U is retractible to W_n^2 by deformation, it is acyclic over Z_m .

A similar construction by Conner and Floyd [4] gives the semi-free action in (e), §9. We retain the notation of the preceding paragraph. If the segment $x \circ y$ in $C \circ C$ is subdivided into three segments of equal length, there is a unique piecewise linear mapping of $x \circ y$ onto the 1-chain $(\{m^k\}x) \circ y - x \circ y + x \circ \{n^k\}y$, and the totality of these mappings for all segments $x \circ y$ gives a mapping $\{m, n\}: C \circ C \rightarrow C \circ C$ which is of degree $m - 1 + n$. Thus $\{m, n\}$ is of degree 0 if $m + n = 1$ as we now suppose. On the semi-infinite cylinder $W^4 = (C \circ C) \times [0, \infty)$, the sets $(\{m^{[t]}, n^{[t]}\}^{-1}c, t)$ form a decomposition of W^4 defining an equivalence ϵ_{mn} . Consider the subsets $W(s, t)$ etc. of W_{mn}^4 . For $\beta = 0, 1, \dots$, $W[\beta, \beta+1]$ is retractible to $W[\beta+1]$ by a deformation which maps $W(t)$, $t \in [\beta, \beta+1)$ onto $W(\beta+1)$. Now

$W(t)$ and $W(k+1)$ are canonically homeomorphic to $C \circ C$ and the maps $W(t) \rightarrow W(k+1)$ are readily identifiable with $\{m, n\}$ and hence are of degree 0. It follows that for $k=0, 1, \dots$, $W[k, k+1]$ is deformable on $W[k, k+1]$ to a point and one can infer from this that W_{mn}^4 is homotopically trivial, hence contractible. Now let p, q be distinct primes. Then the integers can be chosen congruent to 1 modulo p and q respectively (still with $m+n=1$). In this case the transformation $(c, t) \rightarrow (\rho_p \circ \rho_q)c, t$ of W^4 of period pq induces an action (Z_{pq}, W_{mn}^4) which is semi-free and, as in the preceding paragraph, this leads to a semi-free orthogonal action (Z_{pq}, U) where U is contractible (and $\dim U$ is a multiple of pq).

QUESTION. What is the smallest dimension U can have in such an action?

11. Orbit spaces. Let (G, X) be an effective action. Floyd showed [5; 6] that if X is acyclic, one can expect the orbit space X/G to be acyclic. More specifically, if X is a locally compact Hausdorff space and if the compactly supported integral cohomology of X is trivial, the same is true of X/G .

In general, however, the computation of the cohomology of X/G appears to be complicated. Let p be a prime. In the following proposition we denote by $P(X, t)$ the Poincaré polynomial of X for compactly supported cohomology over Z_p and by $Q(a, b)$ the polynomial $t^a + t^{a+1} + \dots + t^b$.

Let $G = Z_p \times Z_p$ and let X be a homology n -sphere over Z_p , p a prime, and let G act effectively on X . Let $Z_p^i, i=1, \dots, p+1$, be the non-trivial cyclic subgroups of G and let $n_i = \dim_p F(Z_p^i), i \geq 1$ and $n_0 = \dim_p F(G)$. The Z_p -cohomology of the orbit space of the free part of the action is given by

$$(1) \quad (1-t)P(X^f/G, t) = \sum_{i=1}^{p+1} Q(n_0+2, n_i+1) - Q(n_0+2, n+1)$$

which can be solved explicitly:

$$(2) \quad P(X^f/G, t) = t^{n_0+2} \sum_{i=2}^{p+1} Q(0, q_1 + \dots + q_{i-1} - 1) Q(0, q_i - 1)$$

where $q_i = n_i - n_0$. Putting $t=1$ in (2) gives

$$(3) \quad \sum_{i=1}^{p+1} (n_i - n_0) = n - n_0.$$

A formula like (1) for higher powers of Z_p has apparently not yet been obtained although a recursion formula has been conjectured

[23]. Borel [18] however showed that the dimensional relations (3) hold with n_i , $i \geq 1$ defined to be $\dim_p F(G^i)$ where G^1, G^2, \dots are the subgroups of index p .

AN APPLICATION. Let (Z_2, P_n) be an effective action where P_n is projective n -space. Then $F(Z_2)$ is empty or else has two components A_1 and A_2 where A_i is a homology projective n_i -space over Z_2 (i.e. $H^*(A_i, Z_2) = H^*(P_{n_i}, Z_2)$), and $n_1 + n_2 = n - 1$.

Suppose in fact that $F(Z_2) \neq \emptyset$. Then there exists a covering action (Z_2, P_n) on the universal covering $\tilde{P}_n = S_n$ of P_n (cf. the proof in §8) such that $g\tilde{x}$ covers gx when \tilde{x} covers x . Let g be the nontrivial element of Z_2 . The transformation $x \rightarrow gx$ permutes with the deck-transformation $t: P_n \rightarrow P_n$ and hence g, t define an action $(Z_2 \times Z_2, P_n)$ whose fixed point sets we denote by \tilde{F} . Of course $\tilde{F}(t) = \emptyset$ so $\tilde{F}(Z_2 \times Z_2) = \emptyset$. Then $\tilde{F}(Z_2 \times Z_2)$, $\tilde{F}(t)$, $\tilde{F}(g)$, $\tilde{F}(tg)$ are homology spheres over Z_2 of dimensions $-1, -1, n_1, n_2$ respectively and by the preceding proposition we have $n_1 + n_2 = n - 1$. Now if \tilde{x} in P_n is fixed under g , then $\phi^{-1}x = (\tilde{x}, t\tilde{x})$ (where ϕ is the projection $\tilde{P}_n \rightarrow P_n$) is invariant under g so that either $g\tilde{x} = \tilde{x}$ or $gt\tilde{x} = \tilde{x}$. It follows that $F(g) = \phi(\tilde{F}(g) \cup \tilde{F}(gt))$. Now $\tilde{F}(g) \cap \tilde{F}(gt) \subset \tilde{F}(ggt) = \tilde{F}(t) = \emptyset$ and in fact $t\tilde{F}(g) \cap \tilde{F}(gt) = \emptyset$ since $t\tilde{F}(g) = \tilde{F}(g)$. Hence $F(g)$ in the disjoint union of $\phi\tilde{F}(g)$ and $\phi\tilde{F}(gt)$, which may be regarded as orbit spaces of free actions of Z_2 on the homology spheres $\tilde{F}(g)$, $\tilde{F}(gt)$ of dimensions n_1 and n_2 , hence have the stated cohomology.

QUESTION. Is this proposition true if it is assumed only that P_n is a homology projective space over Z_2 ?

REFERENCES

1. R. H. Bing, *A homeomorphism between the 3-sphere and the sum of two solid horned spheres*, Ann. of Math. vol. 56 (1952) pp. 354–362.
2. A. Borel, *Nouvelle démonstration d'un théorème de P. A. Smith*, Comment. Math. Helv. vol. 29 (1955) pp. 27–39.
3. H. Cartan and S. Eilenberg, *Homological algebra*, Princeton, Princeton University Press, 1956.
4. P. E. Conner and E. E. Floyd, *On the construction of periodic maps without fixed points*, Proc. Amer. Math. Soc. vol. 10 (1959) pp. 354–360.
5. E. E. Floyd, *Orbit spaces of finite transformation groups. I*, Duke Math. J. vol. 20 (1953) pp. 563–567.
6. ———, *Orbit spaces of finite transformation groups. II*, Duke Math. J. vol. 22 (1955) pp. 33–38.
7. ———, *Fixed point sets of compact abelian Lie groups of transformations*, Ann. of Math. vol. 66 (1957) pp. 30–35.
8. E. E. Floyd and R. W. Richardson, *An action of a finite group on an n -cell without stationary points*, Bull. Amer. Math. Soc. vol. 65 (1959) pp. 77–79.
9. J. J. Greever, *Fixed points of finite transformation groups*, Dissertation, University of Virginia.

10. M. W. Hirsch and Stephen Smale, *On involutions on a 3-sphere*, Notices, Amer. Math. Soc. vol. 6 (1959) pp. 148-149.
11. S. Lefschetz, *Algebraic topology*, Amer. Math. Soc. Colloquium Publications, vol. 27, 1942.
12. B. von Kérekjártó, *Über die endlichen topologischen Gruppen der Kugelfläche*, Nederl. Akad. Wetensch. Proc. Ser. A vol. 22 (1919) pp. 568-569.
13. J. Milnor, *Groups which act on S_n without fixed points*, Amer. J. Math. vol. 79 (1957) pp. 623-630.
14. D. Montgomery and H. Samelson, *A theorem on fixed points of involutions in S^3* , Canad. J. Math. vol. 7 (1955) pp. 208-220.
15. C. D. Papakyriakopoulos, *On Dehns' lemma and the asphericity of knots*, Ann. of Math. vol. 66 (1956) pp. 1-26.
16. G. de Rham, *Sur les nouveaux invariants topologiques de M. Reidemeister*, Recueil Math. (Moscow) new series, vol. 1 (1936) pp. 737-742.
17. H. Seifert and W. Threlfall, *Topologische Untersuchung der Diskontinuitätsbereiche endlicher Bewegungsgruppen des dreidimensionalen sphärischen Raumes. I*, Math. Ann. vol. 104 (1930) pp. 1-70.
18. *Seminar on transformation groups*, Institute for Advanced Study, 1958-195.
19. P. A. Smith, *Transformations of finite period*, Ann. of Math. vol. 39 (1937) pp. 137-164.
20. ———, *Transformations of finite period. II*, Ann. of Math. vol. 40 (1939) pp. 690-711.
21. ———, *Fixed point theorems for periodic transformations*, Amer. J. Math. vol. 63 (1941) pp. 1-8.
22. ———, *Permutable periodic transformations*, Proc. Nat. Acad. Sci. U.S.A. vol. 30 (1944) pp. 105-108.
23. ———, *Orbit spaces of abelian p -groups*, Proc. Nat. Acad. Sci. U.S.A. vol. 45 (1959) pp. 1772-1775.
24. R. G. Swan, *Groups with periodic cohomology*, Bull. Amer. Math. Soc. vol. 65 (1959) pp. 368-370.
25. J. H. C. Whitehead, *An involution of spheres*, Ann. of Math. vol. 66 (1957) pp. 27-29.
26. R. L. Wilder, *Topology of manifolds*, Amer. Math. Soc. Colloquium Publications, vol. 32, 1949.
27. H. Zassenhaus, *Theory of groups*, New York, Chelsea Publishing Co., 1949.
28. ———, *Über endliche Fastkörper*, Abh. Math. Sem. Univ. Hamburg vol. 11 (1936) pp. 187-220.

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