ISOMETRIES OF ORLICZ SPACES

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The purpose of the present note is to sketch a solution for the problem of determining the form of all isometries of any reflexive Orlicz space.[1] A partial result in that direction was obtained earlier by J. Lamperti [4] (who suggested this problem to us recently). The ideas of the proof are very closely related to those used recently by the author to develop a unified and slightly extended theory (unpublished) [6] for the classical results of Banach [1], Stone [8] and Kadison [2] (see also [4]) on isometries of $C(X)$, L_p spaces, and C^* algebras. The systematic use of semi-inner-product spaces, and generalized hermitians [5], plays a central role. A semi-inner-product space, is a vector space X on which there is defined a (complex valued) form $[x, y]$ satisfying:

(i) Linearity in x,

(ii) $[x, x] > 0$ if $x \neq 0$,

(iii) $|[x, y]|^2 \leqq [x, x][y, y]$.

X is then normed under $\|x\| = [x, x]^{1/2}$.

From now on, X is a reflexive Orlicz space [7; 3] whose unit sphere is the set $\{f \in X: \int \phi(|f|) \leqq 1\}$. It is somewhat laborious but not very difficult to show that the semi-inner-product for X is given by:

$$[f, g] = C(g) \int_\Omega f\phi'\left(\frac{|g|}{\|g\|}\right) \operatorname{sgn} g$$

where

$$\operatorname{sgn} g = \begin{cases} \dfrac{|g|}{g} & \text{if } g \neq 0, \\ 0 & \text{if } g = 0 \end{cases}$$

with $C(g) = (\int g\phi'(|g|/\|g\|) \operatorname{sgn} g)^{-1}\|g\|^2$, when g is such that the measure of $\{\xi \in \Omega: \phi$ has no derivative at the point $|g(\xi)|/\|g\|\}$ is 0.

A bounded hermitian operator (see [5]) satisfies by definition $[Hf, f] = $ real for all $f \in X$.

PROPOSITION 1. *If h is real valued and in $L_\infty(\Omega)$, $Hf = hf$ defines a hermitian operator on X, and $\|H\| = \|h\|_\infty$.*

[1] Actually the proof sketched below covers the Orlicz spaces over measure spaces containing no atoms. If the measure space contains atoms, further argument is needed.

THEOREM 2.[2] *If X is different from $L_2(\Omega)$, H is a bounded hermitian on X, then there is a real valued $h \in L_\infty(\Omega)$ such that $Hf = hf$ for all $f \in X$, and $\|H\| = \|h\|_\infty$.*

SKETCH OF THE PROOF. If u and v are in X, and have disjoint supports, Ω_1 and Ω_2, then $\operatorname{Im}[H(e^{i\alpha}u + e^{i\beta}v),\ e^{i\alpha}u + e^{i\beta}v] = 0$. α, β real and arbitrary lead to.

$$\int_{\Omega_2} Hu\phi'(|v|/\|v\|)\operatorname{sgn} v = \left\{\int_{\Omega_1} Hv\phi'(|u|/\|u\|)\operatorname{sgn} u\right\}.$$

One applies this to $u_2 = \alpha\chi_{\Omega_1}$, $u_3 = \beta\chi_{\Omega_1}$, $u_1 = (\alpha+\beta)\chi_{\Omega_1}$ and $v = \chi_{\Omega_2}/\|\chi_{\Omega_2}\|$, where χ_Ω denotes the characteristic function of the measurable set Ω. One arrives finally at:

$$\left\{\phi'\left(\frac{\alpha+\beta}{\lambda_1}\right) - \frac{\phi'\left(\frac{1}{\lambda_1}\right)}{\phi'\left(\frac{1}{\lambda_2}\right)}\phi'\left(\frac{\alpha}{\lambda_2}\right) - \frac{\phi'\left(\frac{1}{\lambda_1}\right)}{\phi'\left(\frac{1}{\lambda_3}\right)}\phi'\left(\frac{\beta}{\lambda_3}\right)\right\}\int_{\Omega_1} Hv = 0$$

where $\lambda_1 = \|u_1 + u_2 + v\|$, $\lambda_2 = \|u_1 + v\|$, $\lambda_3 = \|u_2 + v\|$, $\alpha, \beta > 0$ arbitrary Ω_1, Ω_2 and v fixed. Letting the measure of Ω_1 tend to 0 in a convenient manner λ_1, λ_2 and λ_3 tend to $\|v\| = 1$, so that either $\phi'(\alpha+\beta) = \phi'(\alpha) + \phi'(\beta)$ (i.e., $\phi(\alpha) = k\alpha^2$ and X is $L_2(\Omega)$) or else Hv is 0 on Ω_1. From this follows that if $f \in X$ is a step function and Ω_0 the support of one step, $H(f - f(\Omega_0)1)$ is 0 on Ω_0, hence $Hf = hf$, where $h = H1$. The rest is immediate. From this we obtain the main theorem.

THEOREM 3. *If U is an isometry from X onto X, then it is of the form* $Uf(\cdot) = u(\cdot)f(T\cdot)$ *where T is a measurable transformation in Ω and u a fixed function in X, unless X is a Hilbert space.*

SKETCH OF THE PROOF. The expression $[f, g]' = [Uf, Ug]$ is again a semi-inner-product on X, so that if H is hermitian the same holds for UHU^{-1}. If the real-valued function $h \in L_\infty(\Omega)$, denote by H_h the multiplication operation defined by h (which is hermitian). $UH_hU^{-1} = H_{\hat{h}}$, where $\|\hat{h}\|_\infty = \|H\| = \|h\|_\infty$. Since $UH_hU^{-1}UH_kU^{-1} = UH_{hk}U^{-1}$, the operation ^ is multiplicative, and step functions go into step functions. This defines T; the rest goes smoothly.

REMARK. The previous argument could be modified so as to hold for a form not satisfying condition (iii), if a sufficiently strong condi-

[2] From a letter I received recently from Dr. C. A. McCarthy, it appears that McCarthy had a proof of Theorem 2.

tion is assumed with respect to ϕ. The space would not be an Orlicz space, but an extension of the L_p space for $p<1$. For the latter L_p spaces, it is known that the isometries are as described above.

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ON THE RECURRENCE OF SUMS OF RANDOM VARIABLES

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We give a very short proof of the recurrence theorem of Chung and Fuchs [1] in one and two dimensions. This new elementary proof does not detract from the old one which uses a systematic method based on the characteristic function and yields a satisfactory general criterion. But the present method, besides its brevity, also throws light on the combinatorial structure of the problem.

Let N denote the set of positive integers, M that of positive real numbers. Let $\{X_n,\ n\in \mathrm{N}\}$ be a sequence of independent, identically distributed real-valued random vectors, and let $S_n=\sum_{\nu=1}^{n} X_\nu$. The value x is possible iff for every $\epsilon>0$ there exists an n such that $P\{|S_n-x|<\epsilon\}>0$; it is recurrent iff for every $\epsilon>0$, $P\{|S_n-x|<\epsilon$ for infinitely many $n\}=1$. It is shown in [1] that every possible value is recurrent if and only if for some $m\in \mathrm{M}$ we have

$$(1) \qquad \sum_{n=1}^{\infty} P\{|S_n| < m\} = \infty .$$

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