Rings of continuous functions. By Leonard Gillman and Meyer Jerison. D. Van Nostrand Co., Inc., Princeton, 1960. ix+300 pp. $8.75.

For a completely regular topological space X, let $C(X)$ denote the set of all real-valued continuous functions on X and $C^*(X)$ the set of all bounded functions in $C(X)$. The importance of functions in $C^*(X)$ in the theory of topological spaces was recognized by some of the earliest workers in this field. During the 1920's, P. S. Uryson, P. S. Aleksandrov, and above all A. N. Tihonov studied and exploited functions in $C^*(X)$. E. Čech's construction of βX for completely regular X relies upon *all* functions in $C^*(X)$, as every construction of βX in the end must do. The structure of $C^*(X)$ as an algebra under pointwise operations and its rôle in characterizing X [$C^*(X)$ determines X if and only if X is a compact Hausdorff space] were studied by M. H. Stone in 1937.

Algebras $C^*(X)$ are exactly as complicated to study as are compact Hausdorff spaces, since $C^*(X)$ is isomorphic with $C^*(\beta X)$ for completely regular X, and compact Hausdorff spaces X are determined by $C^*(X)$. No reasonable person who has seriously looked at βR or βN [R denotes the real numbers with the usual topology, N the positive integers with the discrete topology] will claim that $C^*(R)$ or $C^*(N)$ has been fully analyzed. Still the problems appear to be topological rather than algebraic. For a compact Hausdorff space X, every maximal ideal in $C^*(X)$ has the form $M_p = \{f \in C^*(X) : f(p) = 0\}$ for some $p \in X$. Every homomorphism of $C^*(X)$ onto $C^*(X)/M_p$ has the form $f \rightarrow f(p)$. Every ideal in $C^*(X)$ closed in the topology induced by the uniform norm is the intersection of maximal ideals and has the form $\{f \in C^*(X) : f(A) = 0\}$, where A is a closed subset of X. Every closed subalgebra of $C^*(X)$ containing 1 has the form $C^*(Y)$, where Y is a Hausdorff continuous image of X. There is also a neat abstract characterization of $C^*(X)$: a real commutative Banach algebra A with a multiplicative identity u such that $(x^2 - \|x\|^2 u)^{-1}$ exists for no $x \in A$ and $(x^2 + u)^{-1}$ exists for all $x \in A$ is identifiable with $C^*(X)$ for a compact Hausdorff space X. One may say that the theory of $C^*(X)$ has a happy ending.

The algebra $C(X)$ is very different. If $C(X) \neq C^*(X)$, then $C(X)$ is not normable in any fashion. Its maximal ideals M can be put into one-to-one correspondence with the points of βX, but the residue class fields $C(X)/M$ can be [and some are] formidably complicated; a complete analysis of these fields, in spite of recent progress by N. L. Alling and others, seems exceedingly difficult. The most natural topology for $C(X)$ seems to be the m-topology, in which neighborhoods of

0 are the sets $\{f: f\in C(X), |f(x)| < \pi(x) \text{ for all } x\in X\}$, where π is an everywhere positive function in $C(X)$. Under the m-topology $C(X)$ is a topological ring but not a topological algebra [unless $C(X) = C^*(X)$]. Every maximal ideal in $C(X)$ is m-closed and every m-closed ideal is the intersection of maximal ideals; but no characterization of m-closed ideals seems to be known. Subalgebras of $C(X)$ have been studied hardly at all. No really satisfactory abstract characterization of $C(X)$ has been found.

The first paper known to the reviewer dealing with $C(X)$ *per se* was published in 1939 [Dokl. Akad. Nauk SSSR] by Gel'fand and Kolmogorov. The writer of the present review took up the subject in 1946, publishing his results in Trans. Amer. Math. Soc. in 1948. Since that time the ring $C(X)$, topological questions arising from its study, its ideals of various kinds, its homomorphic images, etc., have been studied by a large number of writers. The subject has come of age, and it is thoroughly appropriate for a volume dealing with $C(X)$ to appear at the present time.

Professors Gillman and Jerison have set forth in exemplary fashion the present theory of $C(X)$. The book has been written by painstaking craftsmen, who have given attention to every detail of notation and terminology, who have carefully considered their readers' backgrounds,[1] and who have lavished care and displayed great skill in presenting the best proofs they could devise for every theorem. The book is not written in Landau's lapidary style or anything approaching it. It reminds the writer somehow of a beautifully constructed piece of cabinetwork.

The book contains 16 chapters. Each chapter contains a collection of problems, which range from trivial to fiendishly difficult, and which embody among much else what may well be the world's finest collection of weird examples of topological spaces. For each chapter there are notes, partly historical and partly expansions on the main text. The book contains a sensible bibliography; every title listed is referred to in the text. The index is remarkably complete. Chapter 1, *Functions on a topological space*, Chapter 2, *Ideals and Z-filters*, and Chapter 3, *Completely regular spaces*, are introductory and are largely previously known material, beautifully organized. Chapter 4, *Fixed ideals. Compact spaces*, disposes quickly of the compact case. Chapter 5, *Ordered residue class rings*, singles out and studies the ideals in

[1] "This book is addressed to those who know the meaning of each word in the title: none is defined in the text . . . . The reader is presumed to have some background in general topology and abstract algebra, to the extent, at least, of feeling at home with the basic concepts."

$C(X)$ for which $C(X)/I$ is a partially ordered ring; prime and maximal ideals in particular are discussed. Chapter 6, *The Stone-Čech compactification* and Chapter 7, *Characterization of maximal ideals*, construct, illustrate by examples, and apply to maximal ideals in $C(X)$ the famous space βX of Stone and Čech. Chapter 8, *Realcompact spaces*, deals with the spaces X for which $C(X)/M$ is isomorphic to R only for ideals $\{f \in C: f(p)=0\}$ $(p \in X)$. These spaces, originally christened "Q-spaces" by the reviewer, bear much the same relation to $C(X)$ that compact spaces bear to $C^*(X)$. This chapter also contains a construction of the space vX, which is a realcompact extension of X bearing the same relation to $C(X)$ that βX bears to $C^*(X)$. Chapter 9, *Cardinals of closed sets in* βX, contains the computations indicated and applications to various structure problems. Chapter 10, *Homomorphisms and continuous mappings*, examines the relations between homomorphisms of $C(X)$ into $C(Y)$ and continuous mappings Y into X. In Chapter 11, *Embedding in products of real lines*, βX and vX are obtained by embedding the space X in a product of closed intervals and lines, respectively. Chapter 12, *Discrete spaces. Nonmeasurable cardinals*, contains a discussion of realcompact discrete spaces. Chapter 13, *Hyper-real residue class fields*, gives a detailed analysis of the fields $C(X)/M$ that are not isomorphic with R. [Some subsequent results have been obtained by N. L. Alling.] Chapter 14, *Prime ideals*, deals with totally ordered residue class rings. Chapter 15, *Uniform spaces*, takes up realcompact spaces and the space vX from the point of view of uniform structures. Chapter 16, succinctly titled *Dimension*, relates the dimension of X with properties of $C^*(X)$, the culminating result being Katětov's theorem.

To compensate for the obvious disgrace of delaying a review for 18 months after publication, there are two advantages for the reviewer: one can see what more punctual reviewers have written, and one can also take soundings among one's friends. The reviews in Mathematical Reviews [Volume 22 (1961), pp. 1190–1191] and The American Mathematical Monthly [Volume 68 (1960), p. 519] speak for themselves.[2] The reception of the book by the mathematical public has been enthusiastic. Students of set-theoretic topology everywhere are using the book: the reviewer has seen dog-eared copies in

[2] To the present reviewer, some of the strictures in the first-cited review appear unjustified. A part of topological analysis like the theory of $C(X)$ can surely be judged on its own merits and not merely for its applications to other parts of mathematics. Indeed, the Mathematical Reviews reviewer has given an excellent description of the evolution of a mathematical discipline. *Mutatis mutandis*, his words would apply to the theory of groups or the theory of linear operators.

five countries. Plainly Professors Gillman and Jerison have written a standard work, which will undoubtedly be apotheosized with the passage of time to the status of a classic.

EDWIN HEWITT

Theory of Markov processes. By E. B. Dynkin (translated from the Russian by D. E. Brown, edited by T. Kovary). Prentice-Hall Inc., Englewood Cliffs, N. J. and Pergamon Press, Oxford-London-Paris, 1961. 9+210 pp. $11.95.

Die Grundlagen der Theorie der Markoffschen Prozesse. By E. B. Dynkin. (Die Grundlehren der Mathematischen Wissenschaften, Band 108), translated from the Russian by Joseph Wloka. Springer Verlag, Berlin-Göttingen-Heidelberg, 1961. 12+174 pp. D.M. 29.80.

These are translations of the first of two books by the author on Markoff processes. The second, concerning relationships between these processes and semi-groups of linear operators, will appear in Russian shortly.

There has been much work in recent years on continuous time-parameter Markoff processes in an arbitrary state space and on their relationships with certain objects in analysis, but, until now, no one has put forward a single unified framework within which all this research could be carried on. Such a structure is provided here, and its description is carried out in a systematic and skillful manner.

Although the book is formally self-contained, its real prerequisites are a knowledge of general measure theory together with the amount of the theory of Markoff processes to be found in the books of Feller and Gnedenko. Even a reader with this background may find the reading tedious, for while he will be able to follow the formal development, he will get no glimpse of the interesting research to which this extensive measure-theoretic apparatus is appropriate. Thus, the book tends to look like one hundred and seventy pages of preliminaries. A few comments indicating the applications and related research would have added little to the length of the volume and much to the enjoyment of the reader. Viewed, however, as a rigorous survey of the foundations, this book is a complete success.

Chapter One contains a survey of the necessary measure-theoretic facts, including conditional expectation.

In Chapter Two the general concept of a Markoff process is introduced. A transition function is assumed so that rather than having one measure on the sample space we have a family of them, one for