SOLUTION OF THE WARING-GOLDBACH PROBLEM FOR ALGEBRAIC NUMBER FIELDS[1]

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Let P be the field of the rational numbers and K an arbitrary algebraic number field.

1. **Hua's solution for P.** Let us denote by k and s arbitrary natural numbers, by $c(*)$ a positive constant depending on $*$ only, by $\mathfrak{P}^k$ the sequence consisting of 1 and the kth powers of all rational prime numbers, by $s\mathfrak{P}^k$ the s-fold Schnirelmann sum $\mathfrak{P}^k + \cdots + \mathfrak{P}^k$, by $d(s\mathfrak{P}^k)$ the Schnirelmann density of $s\mathfrak{P}^k$, and by $I_{k,s}(m)$ the number of solutions of

$$p_1^k + \cdots + p_s^k = m, \qquad p_j \in \mathfrak{P}^1 \cup \{0\} \qquad (j = 1, \cdots, s).$$

The Waring-Goldbach problem for P was solved by Hua in 1938 since he proved

THEOREM (1P). *For every natural number k there exists a $c_1(k)$ such that for every $s \geqq c_1(k)$ and every natural number m we have $I_{k,s}(m) > 0$.*

Hua also proved the following stronger

THEOREM (2P). *For*

$$s \geqq \begin{cases} 2^k + 1 & \text{in case } 1 \leqq k \leqq 10, \\ k^2(4 \log k + 2 \log\log k + 5) & \text{in case } k > 10 \end{cases}$$

we have

$$I_{k,s}(m) = c_2(m) \frac{\Gamma^s\left(\frac{1}{k}\right) m^{s/k-1}}{\Gamma\left(\frac{s}{k}\right) \log^s m} \left(1 + O\left(\frac{\log\log m}{\log m}\right)\right) \tag{1}$$

and (for the "singular series" $c_2(m)$)

$$c_2(m) < c_3, \tag{2}$$

$$c_2(m) > c_4, \tag{3}$$

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with absolute positive constants c_3 and c_4.

The proof of (1) rests mainly on

(P1) the obvious formula

$$I_{k,s}(m) = \int_0^1 \left(\sum_{p \leq m^{1/k}} e^{2\pi i p^k \alpha} \right)^s e^{-2\pi i m \alpha} d\alpha \qquad [2, p. 93],$$

(P2) the Farey decomposition of the path of integration (basic and supplementary intervals) [2, pp. 84–85],

(P3) an estimate of trigonometrical sums [2, Hilfssatz 7. 13],

(P4) an estimate of trigonometrical sums with primes [2, Satz 10],

(P5) the prime number theorem of Siegel and Walfisz [2, Hilfssatz 7. 14],

(P6) an estimate of complete trigonometrical sums [2, Satz 1]. In the proof of (1), the facts (P3) and (P4) are used to show that the contribution of the integral over the supplementary intervals is contained in the remainder term of (1); (P5) and (P6) are used to show that the contribution of the integral over the basic intervals yields the main term and the remainder term of (1). The proof of (2) uses only

(P7) simple facts on congruences.

The proof of (3) requires besides (P7) also

(P8) a (considerably deeper) theorem on the solvability of

$$x_1^k + \cdots + x_s^k \equiv m \bmod p^t, \quad p \nmid x_1 \cdots x_s.$$

(1) and (2) imply

$$I_{k,s}(m) < c_5(k, s) m^{s/k-1} \log^{-s} m. \tag{4}$$

(4), (P5), and the Schwarz inequality give immediately (the classical idea of Schnirelmann [5])

THEOREM (3P). *For $s \geq c_6(k)$ we have $d(s\mathfrak{P}^k) > 0$.*

Theorem (3P) is equivalent to Theorem (1P).

2. **The solution for arbitrary** K. Let us denote by $\tilde{\mathfrak{P}}^k$ the sequence consisting of 1 and the kth powers of all totally positive prime numbers ϖ of K, by $s\tilde{\mathfrak{P}}^k$ the generalized s-fold Schnirelmann sum $\tilde{\mathfrak{P}}^k + \cdots + \tilde{\mathfrak{P}}^k$ [4, (15)], by $d(s\tilde{\mathfrak{P}}^k)$ the generalized Schnirelmann density of $s\tilde{\mathfrak{P}}^k$ [4, (16)], and by $J_{k,s}(\mu)$ for $\mu \in K$ the number of solutions of

$$\varpi_1^k + \cdots + \varpi_s^k = \mu, \qquad \varpi_j \in \tilde{\mathfrak{P}}^1 \cup \{0\} \qquad (j = 1, \cdots, s).$$

THEOREM (3K). *For* $s \geqq c_7(k, K)$ *we have* $d(s\tilde{\mathfrak{P}}^k) > 0$.

The idea of the proof of Theorem (3K) is the same as for Theorem (3P); one needs direct generalizations (K1), $\cdots$, (K7) of (P1), $\cdots$, (P7) for P to K. Now, (K1) resp. (K2) resp. (K3) resp. (K5) resp. (K6) can be found in [6] resp. [6] resp. [4] resp. [3] resp. [1]. (K4) and (K7) are practically straightforward generalizations of (P4) and (P7). This completes the outline of a proof of Theorem (3K).

(K8) can also be obtained and one arrives at a Theorem (2K) for $J_{k,s}(\mu)$ which is too involved to be stated here.

A generalization to polynomials presents no serious difficulty.

I would like to express my gratitude to Professor Hans Rademacher for drawing my attention to [3].

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