SOME RESULTS ON INVARIANT THEORY

BY S. HELGASON[1]

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1. **Symmetric invariants.** Let V be a finite-dimensional vector space over $\boldsymbol{R}$. Each $X \in V$ gives rise (by parallel translation) to a vector field on V which we consider as a differential operator $\partial(X)$ on V. The mapping $X \to \partial(X)$ extends to an isomorphism of the complex symmetric algebra $S(V)$ over V onto the algebra of all differential operators on V with constant complex coefficients. Let G be a subgroup of the general linear group $\boldsymbol{GL}(V)$. Let $I(V)$ denote the set of G-invariants in $S(V)$ and let $I_+(V)$ denote the set of G-invariants without constant term. The group G acts on the dual space V^* of V by

$$(g \cdot v^*)(v) = v^*(g^{-1} \cdot v), \qquad g \in G, v \in V, v^* \in V^*,$$

and we can consider $S(V^*)$, $I(V^*)$, $I_+(V^*)$. An element $p \in S(V^*)$ (a polynomial function on V) is called *G-harmonic* if $\partial(J)p = 0$ for each $J \in I_+(V)$. Let $H(V^*)$ denote the set of G-harmonic polynomial functions.

Let V^C denote the complexification of V. Suppose B is a nondegenerate symmetric bilinear form on $V^C \times V^C$. If $X \in V^C$ let X^* denote the linear form $Y \to B(X, Y)$ on V. The mapping $X \to X^*$ extends to an isomorphism $P \to P^*$ of $S(V)$ onto $S(V^*)$. If G leaves B invariant then $I(V)^* = I(V^*)$.

We shall use the following notation: If E and F are linear subspaces of the associative algebra A then EF denotes the set of all sums $\sum_i e_i f_i$, $(e_i \in E, f_i \in F)$.

THEOREM 1. *Let B be a nondegenerate symmetric bilinear form on $V \times V$ and let G be a Lie subgroup of* $\boldsymbol{GL}(V)$ *leaving B invariant. Suppose that either* (1) *G is compact and B positive definite or* (2) *G is connected and semisimple. Then*

$$S(V^*) = I(V^*)H(V^*).$$

The case of a compact G was noted independently by B. Kostant. It is a simple consequence of the fact that under the standard strictly positive definite inner product on $S(V^*)$ (invariant under G), the space $H(V^*)$ is the orthogonal complement to the ideal in $S(V^*)$ generated by $I_+(V^*)$. For the noncompact case, let $\mathfrak{g}$ denote the complexification of the Lie algebra of G. It is not difficult to prove that

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each compact real form $\mathfrak{u}$ of $\mathfrak{g}$ leaves invariant a real form W of V^C on which B is strictly positive definite. Now the compact case can be applied to the action of $\mathfrak{u}$ on W.

In the case when G is the orthogonal group $\boldsymbol{O}(n)$ acting on $V=\boldsymbol{R}^n$ then $I(V^*)$ consists of all polynomials in $x_1^2+\cdots+x_n^2$ and $H(V^*)$ consists of all the ordinary harmonic polynomials. Theorem 1 reduces to the classical fact that each $p=p(x_1,\cdots,x_n)$ can be written $p=\sum_k (x^2+\cdots+x_n^2)^k h_k$ where each h_k is harmonic. It is also known (compare Cartan [2, p. 285], Maass [9]) that $H(V^*)$ is in this case spanned by the polynomials $(a_1x_1+\cdots+a_nx_n)^k$ where $a_1,\cdots,a_n \in \boldsymbol{C}$, $a_1^2+\cdots+a_n^2=0$ and $k=0, 1, \cdots$. The following generalization holds:

THEOREM 2. *Let the assumptions be as in Theorem* 1. *Let N_G denote the set of common zeros (in V^C) of the elements in $I_+(V^*)$. Then $H(V^*)$ is the direct sum*

$$H(V^*) = H_1(V^*) + H_2(V^*),$$

where $H_1(V^*)$ is the vector space spanned by the polynomials $(X^*)^k$, $(k=0, 1, 2, \cdots, X\in N_G)$ and $H_2(V^*)$ is the set of G-harmonic polynomials which vanish identically on N_G.

For the case $G=\boldsymbol{O}(n)$ it follows easily from Hilbert's Nullstellensatz that $H_2(V^*)=0$.

2. **Exterior invariants.** Let $\Lambda(V)$ and $\Lambda(V^*)$, respectively, denote the Grassmann algebras over the dual vector spaces V and V^*. Each $X\in V$ induces an antiderivation $\delta(X)$ of $\Lambda(V^*)$ given by

$$\delta(X)\cdot(x_1\wedge\cdots\wedge x_n) = \sum_{k=1}^{n}(-1)^{k+1}x_k(X)(x_1\wedge\cdots\wedge \hat{x}_k\wedge\cdots\wedge x_n)$$

where $\hat{x}_k$ indicates omission of x_k. The mapping $X\rightarrow\delta(X)$ extends uniquely to an *isomorphism* of $\Lambda(V)$ into the algebra of all endomorphisms of $\Lambda(V^*)$. Let G be any subgroup of $\boldsymbol{GL}(V)$. Let $J(V)$ and $J(V^*)$ denote the set of G-invariants in $\Lambda(V)$ and $\Lambda(V^*)$, respectively, $J_+(V)$ and $J_+(V^*)$ the sets of invariants without constant term. An element $p\in\Lambda(V^*)$ is called *G-primitive* if $\delta(J)p=0$ for each $J\in J_+(V)$. Let $P(V^*)$ denote the set of G-primitive elements.

THEOREM 3. *Let the assumptions be as in Theorem* 1. *Then*

$$\Lambda(V^*) = J(V^*)\wedge P(V^*).$$

EXAMPLE. Let E be an n-dimensional Hilbert space over $\boldsymbol{C}$. Considering E as a $2n$-dimensional vector space V over $\boldsymbol{R}$ the unitary group $U(n)$ becomes a subgroup G of the orthogonal group $O(2n)$. Let $Z_k = X_k + iY_k$ $(1 \leqq k \leqq n)$ be an orthonormal basis of E and let $x_1, y_1, \cdots, x_n, y_n$ be the basis of V^* dual to the basis $X_1, Y_1, \cdots, X_n, Y_n$ of V. It is easy to show that the element

$$u = \sum_1^n x_k \wedge y_k \qquad \left(= \frac{i}{2} \sum_1^n z_k \wedge \bar{z}_k \right)$$

and its powers form a basis of $J_+(V^*)$. In view of Theorem 3 each $v \in \Lambda(V^*)$ can therefore be written

$$v = \sum_k u^k \wedge p_k,$$

where each p_k satisfies $\delta(u)p_k = 0$, (compare Weil [**10**, Théorème 3, p. 26]).

3. **Invariants of Weyl groups.** Let $\mathfrak{u}$ be an arbitrary semisimple Lie algebra over $\boldsymbol{R}$ whose adjoint group U is compact. Let θ be an arbitrary involutive automorphism of $\mathfrak{u}$ and let $\mathfrak{u} = \mathfrak{k} + \mathfrak{p}$ be the decomposition of $\mathfrak{u}$ into eigenspaces of θ for the eigenvalue $+1$ and -1 respectively. Let K denote the analytic subgroup of U corresponding to $\mathfrak{k}$. Let $\mathfrak{h}_\mathfrak{p}$ be a maximal abelian subspace of $\mathfrak{p}$ and extend $\mathfrak{h}_\mathfrak{p}$ to a maximal abelian subalgebra $\mathfrak{h}$ of $\mathfrak{u}$. The Weyl group of $\mathfrak{h}$ is defined as the group of linear transformations of $\mathfrak{h}$ induced by the set of elements in U which leave $\mathfrak{h}$ invariant; the Weyl group of $\mathfrak{h}_\mathfrak{p}$ is defined as the group of linear transformations of $\mathfrak{h}_\mathfrak{p}$ induced by the set of elements in K which leave $\mathfrak{h}_\mathfrak{p}$ invariant. Let these groups be denoted by $W(\mathfrak{h})$ and $W(\mathfrak{h}_\mathfrak{p})$ and let $I(\mathfrak{h}^*)$ and $I(\mathfrak{h}_\mathfrak{p}^*)$ denote the corresponding sets of invariant polynomial functions. It is known that $W(\mathfrak{h}_\mathfrak{p})$ can be described as the group of linear transformations of $\mathfrak{h}_\mathfrak{p}$ induced by those members of $W(\mathfrak{h})$ which leave $\mathfrak{h}_\mathfrak{p}$ invariant. Consequently, if the restriction to $\mathfrak{h}_\mathfrak{p}$ of a function f on $\mathfrak{h}$ is denoted by $\bar{f}$, the mapping $f \to \bar{f}$ maps $I(\mathfrak{h}^*)$ into $I(\mathfrak{h}_\mathfrak{p}^*)$.

THEOREM 4. (i) *Suppose* $\mathfrak{u}$ *is a classical compact simple Lie algebra and* θ *any involutive automorphism of* $\mathfrak{u}$. *Then the restriction mapping* $f \to \bar{f}$ *maps* $I(\mathfrak{h}^*)$ *onto* $I(\mathfrak{h}_\mathfrak{p}^*)$.

(ii) *Part* (i) *does not hold in general for the exceptional simple Lie algebras* $\mathfrak{u} = \mathfrak{e}_6, \mathfrak{e}_7, \mathfrak{e}_8$.

(iii) *Let* $Q(\mathfrak{h}^*)$ *and* $Q(\mathfrak{h}_\mathfrak{p}^*)$, *respectively, denote the set of invariant rational functions on* $\mathfrak{h}$ *and* $\mathfrak{h}_\mathfrak{p}$. *Under the restriction mapping* $f \to \bar{f}$, $Q(\mathfrak{h}^*)$ *is mapped onto* $Q(\mathfrak{h}_\mathfrak{p}^*)$.

REMARKS. As $\mathfrak{u}$ and θ are arbitrary, $\mathfrak{k} + i\mathfrak{p}$ is the most general semisimple Lie algebra over $\boldsymbol{R}$. Parts (i) and (ii) above therefore express a property which is shared by all *classical* simple Lie algebras over $\boldsymbol{R}$, yet fails to hold for all simple Lie algebras over $\boldsymbol{R}$. Part (i) is proved by verification using Cartan's classification [**1**] of the root structures of U and of the symmetric space U/K. Since the groups $W(\mathfrak{h}_{\mathfrak{p}})$ and $W(\mathfrak{h})$ are finite groups generated by reflections, $I(\mathfrak{h}_{\mathfrak{p}}^*)$ and $I(\mathfrak{h}^*)$ are polynomial rings, (Chevalley [**4**]). The degrees of the generators can be readily determined from known facts. It is then found that if the space U/K is $\boldsymbol{E}_6/\boldsymbol{F}_4$, $\boldsymbol{E}_7/(\boldsymbol{E}_6\times\boldsymbol{T})$ or $\boldsymbol{E}_8/(\boldsymbol{E}_7\times \boldsymbol{SU}(2))$, the ring $I(\mathfrak{h}_{\mathfrak{p}}^*)$ contains a homogeneous element of degree 3, 4, and 6, respectively, which cannot be obtained from $I(\mathfrak{h}^*)$ by restriction. Part (iii) had been proved independently by Harish-Chandra.

4. **Fundamental functions on quadrics.** Let G be a topological group, H a closed subgroup, G/H the set of left cosets gH with the natural topology. If f is a complex-valued continuous function on G/H and $x\in G$ then f^x denotes the function on G/H given by $f^x(gH)=f(xgH)$ $(g\in G)$. The function f is called *fundamental* (Cartan [**3**, p. 218]) if the vector space V_f over $\boldsymbol{C}$ spanned by the functions f^x $(x\in G)$ is finite-dimensional.

Consider the quadric $C_{p,q}\subset \boldsymbol{R}^{p+q}$ given by the equation

$$Q(X)\equiv x_1^2+\cdots+x_p^2-x_{p+1}^2-\cdots-x_{p+q}^2=1, \qquad (p\geqq 1,\ q\geqq 0).$$

Let $\boldsymbol{O}(p,q)$ denote the group of linear transformations of $\boldsymbol{R}^{p+q}$ leaving Q invariant. The group $\boldsymbol{O}(p,q)$ acts transitively on $C_{p,q}$ and the subgroup leaving $(1,0,\cdots,0)$ fixed is isomorphic to $\boldsymbol{O}(p-1,q)$ so we make the identification

$$C_{p,q}=\boldsymbol{O}(p,q)/\boldsymbol{O}(p-1,q). \tag{1}$$

It is obvious that if $P=P(x_1,\cdots,x_{p+q})$ is a polynomial then the restriction of P to $C_{p,q}$ is a fundamental function. On the other hand we have

THEOREM 5. *Let f be a fundamental function on $C_{p,q}$. Assume $(p,q)\neq(1,1)$. Then there exists a polynomial $P=P(x_1,\cdots,x_{p+q})$ such that*

$$f=P \qquad \text{on } C_{p,q}.$$

REMARKS. 1. The special case $q=0$ (for which $\boldsymbol{O}(p,q)$ is compact) was already proved by Hecke [**6**] and Cartan [**3**].

2. If $p=1$, the denominator in (1) is compact and by use of a compact real form of the complexification of the Lie algebra of $\boldsymbol{O}(1,q)$ this case can be reduced to the case 1. This procedure fails

for $q=1$ because $O(1, 1)$ is not semisimple and the theorem fails to hold for $(p, q)=(1, 1)$ as the example $f(x_1, x_2)=\cosh^{-1}(|x_1|)$ shows. The case $(p, q)=(1, 2)$ was settled by Loewner [8] using special features of the Lobatchefsky plane.

3. By a method of descent the remaining cases can be reduced to the case $x_1^2+x_2^2-x_3^2=1$ (which differs radically from the case $x_1^2-x_2^2-x_3^2=1$ by the noncompactness of the isotropy group). Here one can make use of the special property of the identity component of the group $O(2, 1)$, namely that every representation of it extends to a representation of the corresponding complex subgroup of $GL(3, C)$, (see Harish-Chandra [5]).

4. From Theorem 1 it is clear that the polynomial P can be taken to be an $O(p, q)$-harmonic polynomial, that is a polynomial satisfying the equation

$$\left(\frac{\partial^2}{\partial x_1^2}+\cdots+\frac{\partial^2}{\partial x_p^2}-\frac{\partial^2}{\partial x_{p+1}^2}-\cdots-\frac{\partial^2}{\partial x_{p+q}^2}\right)P=0.$$

It follows that the function f is necessarily a sum of eigenfunctions of the Laplace-Beltrami operator on $C_{p,q}$ (formed by means of the indefinite Riemannian metric on $C_{p,q}$, [7]).

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Massachusetts Institute of Technology